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Zusammenfassung

Seit einigen Jahren besteht ein wachsendes Interesse an der Approximationen von Funktionen, die von vielen Variablen abhängen. Dies ist motiviert durch zahlreiche Anwendungen in der Physik, der Chemie, der Ökonomie und der Informatik. In den meisten Fällen können die Lösungen nicht analytisch angegeben, dafür aber bis zu einem Schwellenwert ε angenähert werden. Die Komplexität (information complexity) ist für uns definiert als die minimale Anzahl an Operationen $n(d,\varepsilon)$, welche benötigt werden, um das d-variate Problem mit einem Fehler kleiner ε zu lösen. Von großem Interesse dabei ist die Abhängigkeit der Anzahl $n(d,\varepsilon)$ von der Dimension d. In der Literatur wird bei vielen multivariaten Problemen vom sogenannten Fluch der Dimension gesprochen. Dass bedeutet, die benötigte Anzahl an Operationen wächst exponentiell mit der Anzahl an Variablen. Zum Beispiel haben viele multivariate Probleme in isotropen Besov-Räumen $B^t_{p,q}$ und Triebel-Lizorkin-Räumen $F^t_{p,q}$ eine optimale Approximationsrate der Art

$$C_{d,1}n^{-t/d} \le e(n,d) \le C_{d,2}n^{-t/d}, \qquad n \in \mathbb{N}\,,$$

für geeignete positive Konstanten $C_{d,i}$, $i = 1, 2$. Falls jetzt $C_{d,1} > C > 0$ für alle d gilt, dann besteht der Fluch der Dimension für das multivariate Problem. Man kann den Fluch der Dimension bannen, indem man entweder eine bessere Approximationsmethode findet oder die Klasse der zu approximierenden Funktionen verkleinert.

In der Approximationstheorie wird seit den frühen sechziger Jahren mit Funktionenräumen dominierender gemischter Glattheit gearbeitet. In der aktuellen Forschung besteht ein wachsendes Interesse an den Komplexitäten solcher Approximationsprobleme im Höherdimensionalen. Der Grund dafür liegt auf der Hand: Funktionenräume dominierender gemischter Glattheit sind deutlich kleiner als vergleichbare isotrope Räume derselben Glattheit. Im Gegensatz zu dem isotropen Fall ist die optimale Approximationsrate der Funktionen aus den Klassen mit dominierender gemischter Glattheit $S^t_{p,q}A$, wobei $A \in \{B, F\}$, gegeben durch:

$$c_{d,1}n^{-t}(\log n)^{\eta_d} \le e(n,d) \le c_{d,2}n^{-t}(\log n)^{\eta_d}, \qquad n \in \mathbb{N},$$

für geeignete positive Konstanten $c_{d,i}$, $i = 1, 2$. Der Exponent des Logarithmus η_d ist hierbei eine nichtnegative reelle Zahl. Man beachte, dass die Hauptrate (Exponent von n) nicht von der Dimension abhängt. Dieses Ergebnis kann man jetzt mit dem oben beschrieben Fall der isotropen Räume vergleichen. Aus diesem Grund besteht eine realistische Hoffnung, dass man Funktionen dieser Klassen $S^t_{p,q}A$ für höhere Dimensionen als im Falle der isotropen Räume approximieren kann. Das Konzept der dominierenden gemischten Glattheit wird nicht nur in der Approximationstheorie bzw. in IBC (Information based complexity) verwendet. An dieser Stelle sei bemerkt, dass einige Probleme sowohl der Quantenchemie als auch der Wirtschafts- und Finanzmathematik mit Funktionenräumen dominierender gemischter Glattheit modelliert werden, vgl. dazu z.B. [44] and [143].

Funktionenräume dominierender gemischter Glattheit wurden zuerst von S.M. Nikol'skij in den frühen sechziger Jahren eingeführt. Im Falle $d = 2$ definierte er den Sobolev-Raum $S^m_pW(\mathbb{R}^2)$ wie folgt: $S^m_pW(\mathbb{R}^2)$ ist die Menge aller Funktionen $f \in L_p(\mathbb{R}^2)$,

so dass

$$\|f|S^m_pW(\mathbb{R}^2)\| := \|f|L_p(\mathbb{R}^2)\| + \Big\|\frac{\partial^m f}{\partial x_1^m}\Big|L_p(\mathbb{R}^2)\Big\| + \Big\|\frac{\partial^m f}{\partial x_2^m}\Big|L_p(\mathbb{R}^2)\Big\| + \Big\|\frac{\partial^{2m} f}{\partial x_1^m \partial x_2^m}\Big|L_p(\mathbb{R}^2)\Big\|$$

endlich ist. Man beachte, dass die gemischte Ableitung $\frac{\partial^{2m} f}{\partial x_1^m \partial x_2^m}$ maßgeblich für die Norm ist (was auch den Grund für die Namensgebung erklärt). Später wurden diese Räume ebenso wie die verwandten Besov-Räume von verschiedenen Autoren wie z.B. Amanov, Besov, Lizorkin, Nikol'skij, Schmeißer und Triebel intensiv untersucht. Für eine systematische Behandlung dieser Räume wird an dieser Stelle auf die Monographien von Amanov [1] bzw. Schmeißer und Triebel [104] verwiesen. Aktuelle Veröffentlichungen zu Räumen dominierender gemischter Glattheit stammen von Vibyral [140], Bazarkhanov [4, 5, 6], Ullrich [137, 138] und Hansen [45]. Eine der bemerkenswertesten Eigenschaften von Besov-Triebel-Lizorkin-Räumen dominierender gemischter Glattheit besteht in dem Vorliegen einer Kreuz-Quasinorm. Falls $f_i \in A^t_{p,q}(\mathbb{R})$ für $i = 1, \ldots, d$ gilt, dann ist $f = f_1 \otimes \ldots \otimes f_d \in S^t_{p,q}A(\mathbb{R}^d)$ und

$$\|\, f\,|S^t_{p,q}A(\mathbb{R}^d)\| = \prod_{i=1}^d \|\, f_i\,|A^t_{p,q}(\mathbb{R})\|\,.$$

In zwei Spezialfällen können die Räume $S^t_{p,q}A(\mathbb{R}^d)$ als Tensorprodukträume univariater Funktionenräume interpretiert werden. Zum einen gilt für die Bessel-Potential-Räume $S^t_pH(\mathbb{R}^d) = S^t_{p,2}F(\mathbb{R}^d)$, $1 < p < \infty$, die Relation

$$S^t_pH(\mathbb{R}^d) = H^t_p(\mathbb{R}) \otimes_{\alpha_p} H^t_p(\mathbb{R}) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} H^t_p(\mathbb{R})$$

zum anderen gilt für die Skala $S^t_{p,p}B(\mathbb{R}^d) = S^t_{p,p}F(\mathbb{R}^d)$

$$S^t_{p,p}B(\mathbb{R}^d) = B^t_{p,p}(\mathbb{R}) \otimes_{\alpha_p} B^t_{p,p}(\mathbb{R}) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} B^t_{p,p}(\mathbb{R})\,.$$

Hierbei bezeichnet α_p die p-nukleare Norm, siehe [106].

Die vorliegende Arbeit verfolgt zwei Ziele. Erstens werden Funktionenräume dominierender gemischter Glattheit in den Kapiteln 1 - 3 studiert. Zweitens werden das asymptotische Verhalten von Weyl- bzw. Bernstein-Zahlen von den Einbettungen des Tensorproduktes von univariaten Sobolev- bzw. Besov-Räumen in Lebesgue-Räume untersucht (Kapitel 4). Besov- und Triebel-Lizorkin-Räume mit dominierender gemischter Glattheit $S^t_{p,q}B(\mathbb{R}^d)$ und $S^t_{p,q}F(\mathbb{R}^d)$ werden hier auf dem Fourier-analytischen Wege eingeführt. Weitere Eigenschaften, z.B. Dualität, komplexe Interpolation und Charakterisierungen mittels iterierter Differenzen, werden im folgenden studiert.

Ausgangspunkt für die Untersuchungen im Kapitel 2 sind die offensichtlichen Einbettungen

$$W^{dm}_p(\mathbb{R}^d) \hookrightarrow S^m_pW(\mathbb{R}^d) \hookrightarrow W^m_p(\mathbb{R}^d), \qquad m \in \mathbb{N}\,, \quad 1 < p < \infty\,.$$

Zentrales Anliegen des Kapitels ist nun die Beantwortung der Frage, unter welchen Bedingungen die folgenden Einbettungen gelten:

$$S^t_{p,q}A(\mathbb{R}^d) \hookrightarrow A^t_{p,q}(\mathbb{R}^d) \qquad \text{und} \qquad A^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}A(\mathbb{R}^d)\,.$$

Unter Verwendung Fourierscher Multiplikatoren sowie spezieller Testfunktionen war es uns möglich, die Parameterkonstellationen für die Gültigkeit der Einbettungen $S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow B^t_{p,q}(\mathbb{R}^d)$ bzw. $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$ zu charakterisieren. Ähnliche Ergebnisse wurden für Triebel-Lizorkin-Räume unter den Einschränkungen $1 < p < \infty$ und $1 \le q \le \infty$ erzielt. Darüber wird die Optimalität dieser Einbettungen aus verschiedenen Blickwinkeln diskutiert.

Als Konsequenz der Kreuznorm-Eigenschaft der Sobolev-Räume mit dominierender gemischter Glattheit und dem Fakt, dass die Sobolev-Räume $W^m_p(\mathbb{R})$, $1 < p < \infty$, Multiplikationsalgebren sind, folgt, dass es eine Konstante $C > 0$ gibt, so dass

$$\| f \cdot g \,|S^m_p W(\mathbb{R}^d)\| \le C \, \|f|S^m_p W(\mathbb{R}^d)\| \cdot \| g \,|S^m_p W(\mathbb{R}^d)\|$$

gilt für alle $f = f_1 \otimes \ldots \otimes f_d$ und $g = g_1 \otimes \ldots \otimes g_d$, $f_i, g_i \in W^m_p(\mathbb{R})$, $i = 1, \ldots, d$. Im Abschnitt 3.1 werden wir diese Abschätzung verbessern. Genauer gesagt werden wir die folgende Ungleichung beweisen

$$\| f \cdot g \,|S^m_p W(\mathbb{R}^d)\| \le C_1 \, \| f \,|S^m_p W(\mathbb{R}^d)\| \cdot \|g \,|S^m_p W(\mathbb{R}^d)\|$$

mit einer positiven Konstanten C_1, unabhängig von $f, g \in S^m_p W(\mathbb{R}^d)$. Ein analoges Ergebnis wird für das Tensorprodukt der Besov-Räume bewiesen. Außerdem werden wir eine Charakterisierung der Menge aller punktweisen Multiplikatoren für diese Räume $S^t_{p,p}B(\mathbb{R}^d)$ unter der Einschränkung $t > 1/p$ angeben. Es war doch etwas überraschend für uns, dass die sogenannten Moser-Typ-Ungleichungen, welche für die isotropen Besov-Triebel-Lizorkin-Räume wohlbekannt sind, nicht im Rahmen der Räume mit dominierender gemischter Glattheit gelten. Das heißt, es gibt keine Konstante $C > 0$ so dass

$$\|fg|S^m_p W(\mathbb{R}^d)\| \le C\big(\|f|S^m_p W(\mathbb{R}^d)\| \cdot \|g|L_\infty(\mathbb{R}^d)\| + \|f|L_\infty(\mathbb{R}^d)\| \cdot \|g|S^m_p W(\mathbb{R}^d)\|\big)$$

für alle $f, g \in S^m_p W(\mathbb{R}^d)$ gilt. Als Ergänzung untersuchen wir die punktweisen Multiplikatoren für die Räume, die auf dem Einheitswürfel definiert sind. Die Ergebnisse, welche für die Klassen auf $\mathbb{R}^d$ bewiesen wurden, übertragen sich ohne weitere Komplikationen auf die lokale Situation. Abschließend dazu möchten wir anmerken, dass die Ergebnisse über punktweise Multiplikatoren in Sobolev- und Besov-Räumen nicht nur interessant sind im Rahmen der Theorie der partiellen Differentialgleichungen oder bei der Untersuchung von Abbildungseigenschaften nichtlinearer Superpositionsoperatoren, sondern auch in der Lerntheorie (learning theory); für Details verweisen wir auf [68].

Der Zweck von Abschnitt 3.2 ist die Untersuchung der Beschränktheit von speziellen Operatoren, welche Variablensubstitutionen zugeordnet sind. Sei $\varphi \in C^r_0(\mathbb{R})$, so dass $\operatorname{supp} \varphi \subset [0, 1]$, $\int_0^1 \varphi(\xi)\, \mathrm{d}\xi = 1$, $\varphi(\xi) > 0$ auf $(0, 1)$ und die $r$te Ableitung $\varphi^{(r)}$ nur endlich viele Nullen in $[0, 1]$ hat. Sei $\psi(\xi) := \int_{-\infty}^{\xi} \varphi(s)\, \mathrm{d}s$. Dann zeigen wir, dass der Operator

$$T_\psi \colon \; f(x) \;\mapsto\; \Big(\prod_{i=1}^{d} \varphi(x_i)\Big) f\big(\psi(x_1), \ldots, \psi(x_d)\big), \quad x \in \mathbb{R}^d$$

den Raum $S^t_{p,q}A(\mathbb{R}^d)$ auf sich selbst abbildet, solange $1 < p, q \le \infty$, $t > \frac{1}{p}$ und $r > [t] + 1$ sind. Die Einschränkung $r > [t] + 1$ ist natürlich. Dieses Ergebnis kann man vergleichen mit denen von Bykovskii [12], Temlyakov [120] und Dubinin [25, 26] dazu erzielten Resultaten. Diese Autoren nutzen die Bedingung $r \ge \left[\frac{tp}{p-1}\right] + 1$ bzw. $\varphi \in$

$C_0^\infty(\mathbb{R})$. Durch die Anwendung der Operatoren der Variablensubstitution können wir zeigen, dass das Verhalten des Worst-Case-Fehlers von Kubatur-Formeln für Funktionen von dominierender gemischter Glattheit auf $\Omega := [0,1]^d$ asymptotisch nicht schlechter ist als der Worst-Case-Fehler für die Kubatur-Formel für Funktionen mit Trägern streng innerhalb von Ω. Auf diese Art und Weise erhalten wir die richtige Ordnung des Worst-Case-Integrationsfehlers für $S^t_{p,q}A(\Omega)$, das heißt,

$$\mathrm{Int}_n(S^t_{p,q}A(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(1-\frac{1}{q})}, \qquad n \geq 2\,,$$

mit $1 < p, q \leq \infty$ ($p < \infty$ im F-Fall).

Das letzte Kapitel 4 beschäftigt sich mit der Untersuchung des asymptotischen Verhaltens von Weyl- und Bernstein-Zahlen von Einbettung des Tensorproduktes der Sobolev- und Besov-Räume in Lebesgue-Räume auf dem Einheitswürfel. Dies ist der zentrale Gegenstand dieser Dissertation. Das besondere Interesse am Verhalten der Weyl-Zahlen ergibt sich aus der Tatsache, dass sie die kleinsten bekannten s-Zahlen sind, für welche die berühmte Weyl-Typ-Ungleichung erfüllt ist. Sei $T : X \to X$ ein kompakter linearer Operator in einem Banachraum X und $\{\lambda_n(T)\}_{n=1}^\infty$ die Folge der angeordneten, von Null verschiedenen Eigenwerte von T. Jeder Eigenwert wird entsprechend seiner algebraischen Vielfachheit wiederholt und wir verlangen $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|$, $n \in \mathbb{N}$. Dann gilt für alle $n \in \mathbb{N}$

$$\Big(\prod_{k=1}^{2n-1} |\lambda_k(T)|\Big)^{1/(2n-1)} \leq \sqrt{2e}\,\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}\,, \tag{Z.1}$$

siehe Pietsch [85] und Carl, Hinrichs [17]. Die Möglichkeit, das Verhalten von Eigenwerten über Abschätzungen von Weyl-Zahlen zu kontrollieren, macht die Bedeutung der Weyl-Zahlen aus. Oftmals kann man Operatoren schreiben als eine Hintereinanaderausführung einer geeigneten Identität und eines weiteren beschränkten Operators, siehe beispielsweise die Monographien von König [55] und von Edmunds, Triebel [33]. Dies motiviert die Untersuchung von Weyl-Zahlen bzgl. Identitäten.

Wir untersuchen in diesem Kapitel das Verhalten der Weyl-Zahlen $x_n(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega))$, wobei $q \in \{p_0, 2\}$ und $\Omega = [0,1]^d$ ist (zur Erinnerung $S^t_{p_0,2}F(\Omega) = S^t_{p_0}H(\Omega)$ und $S^t_{p_0,p_0}F(\Omega) = S^t_{p_0,p_0}B(\Omega)$). Die benutzte Beweismethode ist in einem gewissen Sinne Standard. Mit Hilfe von Wavelet-Isomorphismen reduzieren wir die Betrachtung $x_n(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega))$ auf die Untersuchung der Zahlen $x_n(id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f)$. Hierbei bezeichnen $s^{t,\Omega}_{p_0,q}f$ bzw. $s^{t,\Omega}_{p,2}f$ entsprechende Folgenräume. Bezüglich der Abschätzung von oben besteht der Beweis in einer Reduktionstechnik. Wir teilen id^* in $id^* = \sum_{\mu=0}^\infty id^*_\mu$ auf (id^*_μ sind Identitäten bezüglich bestimmter Teilräume), man erhält

$$x_n(id^*) \leq \sum_{\mu=0}^{J} x_{n_\mu}(id^*_\mu) + \sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) + \sum_{\mu=L+1}^{\infty} \|id^*_\mu\|\,,$$

mit $n-1 = \sum_{\mu=0}^{L}(n_\mu - 1)$. Nun besteht das eigentliche Problem in der möglichst cleveren Wahl von J, L und n_μ in Abhängigkeit aller Parameter. In einem weiteren Reduktionsschritt werden dann die Abschätzungen von $x_{n_\mu}(id^*_\mu)$ zurückgeführt auf die Abschätzungen von $x_k(id : \ell^m_{p_0} \to \ell^m_p)$. Bzgl. der Abschätzung von unten besteht das Problem darin, geeignete Teilräume von $s^{t,\Omega}_{p_0,q}f$ ausfindig zu machen und dann auch in diesem Fall die Abschätzungen von $x_k(id : \ell^m_{p_0} \to \ell^m_p)$ zu verwenden. Erfreulicherweise ist das Verhalten der

Weyl-Zahlen $x_k(id: \ \ell_{p_0}^m \to \ell_p^m)$ wohlbekannt, siehe Lubitz [65], König [55] und Caetano [13, 14].

Ergänzend möchten wir anmerken, dass diese Methode in den beiden Extremfällen, welche entweder durch $p = 1$ oder durch $p = \infty$ gegeben sind, nicht hinreichend gut funktioniert. Grund hierfür ist der Fakt, dass L_1 und L_∞ keine Littlewood-Paley-Charakterisierungen erlauben. Zur Überwindung dieses Hindernisses verwenden wir Interpolationseigenschaften von Weyl-Zahlen und eine Beziehung mit absolut $(r, 2)$-summierenden Normen. Damit gelingt es uns auch in den besonders schwierigen Extremfällen, die richtige Konvergenzordnung der Weyl-Zahlen zu bestimmen. Als zusätzlichen Gewinn erhalten wir dabei so ganz nebenbei die scharfe untere Abschätzung für Approximations- und Gelfand-Zahlen dieser Einbettungen, ein Problem, an dem seit den sechziger Jahren des vergangenen Jahrhunderts gearbeitet wurde.

Bernstein-Zahlen wurden zuerst von Mityagin und Pelczynski eingeführt [73], wir verweisen aber auch auf Tikhomirov [124]. In Approximationstheorie dienen Bernstein-Zahlen oft als untere Schranke für Gelfand- und Kolmogorov-Zahlen sowie die nichtlinearen Weiten (nonlinear widths), siehe [22, 83]. Im krassen Unterschied zu Weyl-Zahlen, funktioniert die Zerlegungstechnik nicht für Bernstein-Zahlen, da sie sind keine additiven s-Zahlen darstellen. Bezüglich der Abschätzung von oben zeigen wir, dass Bernstein-Zahlen von Entropiezahlen dominiert werden, das heißt, es gilt

$$b_n(T) \le 2\sqrt{2} e_n(T), \quad n \in \mathbb{N},$$

gültig für jeden linearen und beschränkten Operator T. Daraus und aus der Ungleichung

$$b_{2n-1}(T) \le e\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}, \quad n \in \mathbb{N},$$

kombiniert mit dem bekannten asymptotischen Verhalten der Weyl-Zahlen, erhalten wir die scharfe obere Abschätzung für die Bernstein-Zahlen. Insbesondere können wir zeigen, dass die Ungleichung

$$\begin{aligned} b_n(id: S^t_{p_0,q}F(\Omega) \to L_p(\Omega)) \\ \asymp \min\{x_n(id: S^t_{p_0,q}F(\Omega) \to L_p(\Omega)),\ e_n(id: S^t_{p_0,q}F(\Omega) \to L_p(\Omega))\} \end{aligned}$$

fast immer gilt. Die Ausnahme stellt der Fall $2 < p_0 = q < p < \infty$ dar. Abschließend geben wir noch eine Übersicht zur Literatur, insbesondere vergleichen wir unsere Ergebnisse in diesem Kapitel mit denen von Galeev [40].

Neben Weyl-Zahlen sind Entropiezahlen auch ein gutes Werkzeug, um Eigenwerte von kompakten Operatoren zu kontrollieren. Dies wird illustriert durch die wohlbekannte Carl-Triebel-Ungleichung

$$|\lambda_n(T)| \le \sqrt{2}\, e_n(T)\,. \tag{Z.2}$$

Im Abschnitt 4.6 werden wir das Verhalten der Weyl-Zahlen mit dem der Entropiezahlen vergleichen. Basierend auf diesen Ergebnissen und den Ungleichungen (Z.1), (Z.2) können wir zeigen, wie Weyl-Zahlen und Entropiezahlen die Eigenwerte eines kompakten Operators in $L_p(\Omega)$ kontrollieren. In einigen Situationen ist die Ungleichung (Z.1) besser als (Z.2).

Im letzten Abschnitt dieses Kapitels erinnern wir an einige bekannte Resultate bzgl. des asymptotischen Verhaltens einiger weiterer s-Zahlen, welche eng verknüpft sind mit

Weyl- und Bernstein-Zahlen und eine zentrale Rolle in der Approximationstheorie und in IBC spielen.

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1. V.K. Nguyen, Bernstein numbers of embeddings of isotropic and dominating mixed Besov spaces, Math. Nachr. 288 (2015), 1694–1717.

2. V.K. Nguyen, W. Sickel, Weyl numbers of embeddings of tensor product Besov spaces, J. Approx. Theory 200 (2015), 170–220.

3. V.K. Nguyen, Weyl and Bernstein numbers of embeddings of Sobolev spaces with dominating mixed smoothness, J. Complexity 36 (2016), 46–73.

4. V.K. Nguyen, M. Ullrich, T. Ullrich, Change of variable in spaces of mixed smoothness and numerical integration of multivariate functions on the unit cube, Constr. Approx. (2017), DOI: 10.1007/s00365-017-9371-9.

5. V.K. Nguyen, W. Sickel, Isotropic and dominating mixed Besov spaces - a comparison, Functional Analysis, Harmonic Analysis and Image Processing: A collection of papers in honor of Björn Jawerth, Contemp. Math., AMS, to appear.

6. V.K. Nguyen, Gelfand numbers of embeddings of mixed Besov spaces,J. Complexity 41 (2017), 35-57.

7. V.K. Nguyen, W. Sickel, Pointwise multipliers for Sobolev and Besov spaces of dominating mixed smoothness, J. Math. Anal. Appl. 452 (2017), 62–90.

8. V.K. Nguyen, W. Sickel, Isotropic and dominating mixed Lizorkin - Triebel spaces - a comparison, Analysis Mathematica, to appear.

Prepage

There has been increasing interest in solving problems which involve functions defined on high-dimensional domains. Those problems occur in numerous applications such as physics, chemistry, economics, finance, and computational sciences. In most of the cases, the solutions can not be solved analytically but approximated with a threshold ε. The information complexity is defined as the minimal number $n(d,\varepsilon)$ of information operations needed to solve the d-variate problem with an error less than ε. The question that attracts a lot of attention in computational sciences is how $n(d,\varepsilon)$ depends on the dimension d. In the literature, many multivariate problems suffer from the so-called curse of dimensionality. That means the number of information operations needed increases exponentially in the number of variables. For example, many multivariate problems in isotropic Besov $B^t_{p,q}$ and Triebel-Lizorkin classes $F^t_{p,q}$ have an optimal approximation rate of the form

$$C_{d,1} n^{-t/d} \le e(n,d) \le C_{d,2} n^{-t/d}, \qquad n \in \mathbb{N},$$

for suitable positive constants $C_{d,i}$, $i = 1, 2$. Such result indicates that if $C_{d,1} > C > 0$ for all d, then the multivariate problem suffers from the curse of dimensionality. Normally, to overcome the curse of dimensionality one either finds a better approximation method or shrinks the class of approximated functions.

Function spaces of dominating mixed smoothness have found applications in approximation theory since the early sixties. Recently, there is an increasing interest in information-based complexity and high-dimensional approximation. The reason for this is clear. Function spaces of dominating mixed smoothness are much smaller than their isotropic counterparts (with the same smoothness). Different from the isotropic situation, the optimal rate of approximation of functions from the classes of dominating mixed smoothness $S^t_{p,q}A$, where $A \in \{B, F\}$, usually has the form

$$C_{d,1} n^{-t} (\log n)^{\eta_d} \le e(n,d) \le C_{d,2} n^{-t} (\log n)^{\eta_d}, \qquad n \in \mathbb{N}.$$

The power of logarithm term η_d is a non-negative number. Observe that the main rate (power of n) does not depend on the dimension. This result should be compared with that in the case of isotropic spaces. For this reason, there is a realistic hope that one can approximate functions from these classes for larger dimension than in case of isotropic spaces. The concept of dominating mixed smoothness is not only used in approximation theory and information-based complexity. Let us mention that there exist a number of problems in finance and quantum chemistry modeled on function spaces of dominating mixed smoothness, see, e.g., [44] and [143].

Function spaces of dominating mixed smoothness were first introduced by S.M. Nikol'skij in the early sixties. He defined the space of Sobolev type $S^m_p W(\mathbb{R}^2)$ which is the collection of all functions $f \in L_p(\mathbb{R}^2)$ such that

$$\begin{aligned} \|f|S^m_p W(\mathbb{R}^2)\| := \|f|L_p(\mathbb{R}^2)\| &+ \Big\|\frac{\partial^m f}{\partial x_1^m}\Big|L_p(\mathbb{R}^2)\Big\| \\ &+ \Big\|\frac{\partial^m f}{\partial x_2^m}\Big|L_p(\mathbb{R}^2)\Big\| + \Big\|\frac{\partial^{2m} f}{\partial x_1^m \partial x_2^m}\Big|L_p(\mathbb{R}^2)\Big\| \end{aligned}$$

is finite. Observe that the mixed derivative $\frac{\partial^{2m} f}{\partial x_1^m \partial x_2^m}$ plays the dominant role in this norm which is the reason for the name of these spaces. Later on, this type of spaces as well as related Besov spaces have been studied extensively by many authors such as Amanov, Besov, Lizorkin, Nikol'skij, Schmeißer, and Triebel. For a systematic treatment of those spaces we refer to Amanov [1], Schmeißer and Triebel [104]. Recently spaces of dominating mixed smoothness are studied in the booklet of Vibyral [140], Bazarkhanov [4, 5, 6], Ullrich [137, 138], and Hansen [45]. Probably, one of the most interesting properties of Besov-Triebel-Lizorkin spaces of dominating mixed smoothness consists in the cross-quasi-norm, i.e., if $f_i \in A^t_{p,q}(\mathbb{R})$ for $i = 1, \ldots, d$, then $f = f_1 \otimes \ldots \otimes f_d \in S^t_{p,q}A(\mathbb{R}^d)$ and

$$\| f \,|S^t_{p,q}A(\mathbb{R}^d)\| = \prod_{i=1}^{d} \| f_i \,|A^t_{p,q}(\mathbb{R})\| \,.$$

In particular, the Bessel-potential spaces $S^t_pH(\mathbb{R}^d)$ $(1 < p < \infty)$ which are special cases of Triebel-Lizorkin spaces, i.e., $S^t_pH(\mathbb{R}^d) = S^t_{p,2}F(\mathbb{R}^d)$, and $S^t_{p,p}B(\mathbb{R}^d)$ can be identified with the d-fold tensor product of corresponding isotropic spaces on $\mathbb{R}$, i.e.,

$$S^t_pH(\mathbb{R}^d) = H^t_p(\mathbb{R}) \otimes_{\alpha_p} H^t_p(\mathbb{R}) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} H^t_p(\mathbb{R})$$

and

$$S^t_{p,p}B(\mathbb{R}^d) = B^t_{p,p}(\mathbb{R}) \otimes_{\alpha_p} B^t_{p,p}(\mathbb{R}) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} B^t_{p,p}(\mathbb{R})$$

where $1 < p < \infty$ and α_p denotes the p-nuclear tensor norm. Those results have been proved by Sickel and Ullrich [106].

The aim of this thesis is twofold. First we study function spaces of dominating mixed smoothness which is contained in Chapters 1 - 3. Secondly, we investigate the asymptotic behaviour of Weyl and Bernstein numbers of embedding of tensor product Sobolev and Besov spaces into Lebesgue spaces (Chapter 4). It is the purpose of Chapter 1 to recall Besov-Triebel-Lizorkin spaces of dominating mixed smoothness $S^t_{p,q}B(\mathbb{R}^d)$ and $S^t_{p,q}F(\mathbb{R}^d)$ from the Fourier analytic approach and to study some further properties of these spaces such as duality, complex interpolation, and characterization by iterated differences which are useful tools for later investigation.

Chapter 2 is motivated by the intuitive chain of continuous embeddings

$$W^{dm}_p(\mathbb{R}^d) \hookrightarrow S^m_pW(\mathbb{R}^d) \hookrightarrow W^m_p(\mathbb{R}^d),$$

where $1 < p < \infty$ and $m \in \mathbb{N}$. In this chapter we concentrate on the study under which conditions the following embeddings

$$S^t_{p,q}A(\mathbb{R}^d) \hookrightarrow A^t_{p,q}(\mathbb{R}^d) \qquad \text{and} \qquad A^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}A(\mathbb{R}^d)$$

hold true. By using Fourier multiplier assertions and some special test functions we are able to give a complete answer showing for which values of the parameters t, p, q we have the embeddings $S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow B^t_{p,q}(\mathbb{R}^d)$ and $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$. Beside that we also give some results on when the converse embeddings hold. Similar results are obtained for Triebel-Lizorkin spaces in cases $1 < p < \infty$ and $1 \le q \le \infty$. Moreover, we shall discuss the optimality of these embeddings in various directions.

As a consequence of the cross-norm of Sobolev spaces of dominating mixed smoothness and the multiplication algebra property of $W^m_p(\mathbb{R})$ it follows that there exists a constant $C > 0$ such that

$$\| f \cdot g \,|S^m_pW(\mathbb{R}^d)\| \le C \,\|f|S^m_pW(\mathbb{R}^d)\| \cdot \| g \,|S^m_pW(\mathbb{R}^d)\|$$

holds for all $f = f_1 \otimes \ldots \otimes f_d$ and $g = g_1 \otimes \ldots \otimes g_d$ where $f_i, g_i \in W^m_p(\mathbb{R})$, $i = 1, \ldots, d$. In Section 3.1 we will improve this estimate. More precisely, we shall prove that the following inequality

$$\| f \cdot g \,| S^m_p W(\mathbb{R}^d)\| \le C_1 \| f \,| S^m_p W(\mathbb{R}^d)\| \cdot \| g \,| S^m_p W(\mathbb{R}^d)\|$$

holds for all $f, g \in S^m_p W(\mathbb{R}^d)$. Here $C_1 > 0$ is independent of $f, g \in S^m_p W(\mathbb{R}^d)$. An analogous result is obtained for tensor product Besov spaces. In addition we shall give a characterization of the spaces of all pointwise multipliers for those spaces at least under certain restrictions. It was surprising to us that Moser-type inequalities, which are well-known for isotropic Besov-Triebel-Lizorkin spaces, do not hold in the context of spaces of dominating mixed smoothness, i.e., there exists no constant $C > 0$ such that

$$\|fg|S^m_p W(\mathbb{R}^d)\| \le C\big(\|f|S^m_p W(\mathbb{R}^d)\| \cdot \|g|L_\infty(\mathbb{R}^d)\| + \|f|L_\infty(\mathbb{R}^d)\| \cdot \|g|S^m_p W(\mathbb{R}^d)\|\big)$$

holds for all $f, g \in S^m_p W(\mathbb{R}^d)$. As a supplement we investigate the pointwise multipliers for spaces defined on the unit cube. The results obtained for the spaces on $\mathbb{R}^d$ carry over to the local situation. We would like to remark that the results about pointwise multiplication in Sobolev and Besov spaces are of interest not only in the theory of partial differential equations and the study of nonlinear superposition operators, but also in Learning Theory, e.g., see [68].

The purpose of Section 3.2 is to study the boundedness of change of variable operators in spaces of dominating mixed smoothness. Let $\varphi \in C^r_0(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset [0,1]$, $\int_0^1 \varphi(\xi)\,\mathrm{d}\xi = 1$, $\varphi(\xi) > 0$ on $(0,1)$ and the rth derivative $\varphi^{(r)}$ has only finitely many zeros in $[0,1]$. By putting $\psi(\xi) := \int_{-\infty}^{\xi} \varphi(s)\,\mathrm{d}s$ we then prove that the operator

$$T_\psi\colon\ f(x) \ \mapsto \ \Big(\prod_{i=1}^{d} \varphi(x_i)\Big) f\big(\psi(x_1), \ldots, \psi(x_d)\big), \quad x = (x_1, \ldots, x_d) \in \mathbb{R}^d$$

is bounded from $S^t_{p,q}A(\mathbb{R}^d)$ into itself in case of $1 < p, q \le \infty$, $t > \frac{1}{p}$ and $r > [t] + 1$. The restriction $r > [t] + 1$ is quite natural. This result has to be compared with those of Bykovskii [12], Temlyakov [120] and Dubinin [25, 26] where they need the condition $r \ge \big[\frac{tp}{p-1}\big] + 1$ or $\varphi \in C^\infty_0(\mathbb{R})$. By employing the boundedness of change of variable operators we can show that the behaviour of the worse-case error of cubature formulas for functions of dominating mixed smoothness on $\Omega := [0,1]^d$ does not perform asymptotically worse than the cubature formulas for functions with supports strictly inside Ω. As a consequence of this we obtain the correct order of the worst-case integration errors for $S^t_{p,q}A(\Omega)$, i.e.,

$$\mathrm{Int}_n(S^t_{p,q}A(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(1-\frac{1}{q})}, \quad n \ge 2$$

with $1 < p, q \le \infty$ ($p < \infty$ in the F-case).

The final Chapter 4 is devoted to study of the asymptotic behaviour of Weyl and Bernstein numbers of embeddings of tensor product Sobolev and Besov spaces into Lebesgue spaces on the unit cube. The notion of Weyl numbers has its roots in the study of eigenvalues of compact operators. More precisely, Weyl numbers are the smallest known s-numbers satisfying the famous Weyl-type inequalities. Let $T : X \to X$ be a compact linear operator in a Banach space X and $\{\lambda_n(T)\}_{n=1}^\infty$ be the sequence of non-zero eigenvalues of T, ordered in the following way: each eigenvalue is repeated according to its algebraic multiplicity and $|\lambda_n(T)| \geq |\lambda_{n+1}(T)|$, $n \in \mathbb{N}$. Then the inequality

$$\Big(\prod_{k=1}^{2n-1} |\lambda_k(T)|\Big)^{1/(2n-1)} \leq \sqrt{2e}\,\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n} \tag{P.1}$$

holds for all $n \in \mathbb{N}$, see Pietsch [85] and Carl, Hinrichs [17]. Hence, it shows the importance of estimates of Weyl numbers in the study of eigenvalue distributions of compact operators. Many times operators of interest can be written as a composition of an identity between appropriate function spaces and a further bounded operator, see, e.g., the monographs of König [55] and of Edmunds, Triebel [33]. This motivates the study of Weyl numbers of identity operators.

As already mentioned above, tensor product Sobolev and Besov spaces are special cases of Triebel-Lizorkin spaces of dominating mixed smoothness. Hence, in this chapter we shall study $x_n(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega))$ here $q \in \{p_0, 2\}$ and $\Omega = [0,1]^d$ (since $S^t_{p_0,2}F(\Omega) = S^t_{p_0}H(\Omega)$ and $S^t_{p_0,p_0}F(\Omega) = S^t_{p_0,p_0}B(\Omega)$). In the Littlewood-Paley case, i.e., $1 < p < \infty$, we obtain the complete picture of behaviour of the Weyl numbers of these embeddings except limiting cases. The proof is in some sense standard. By means of wavelet characterizations we switch from the consideration of $id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega)$ to $id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f$, where $s^{t,\Omega}_{p_0,q}f$ and $s^{t,\Omega}_{p,2}f$ are appropriate sequence spaces. Concerning the estimate from above the proof consists in a reduction technique. We split id^* into $id^* = \sum_{\mu=0}^{\infty} id^*_\mu$ (id^*_μ are identities with respect to certain subspaces) which results in an estimate

$$x_n(id^*) \leq \sum_{\mu=0}^{J} x_{n_\mu}(id^*_\mu) + \sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) + \sum_{\mu=L+1}^{\infty} \|id^*_\mu\|\,,$$

and $n - 1 = \sum_{\mu=0}^{L}(n_\mu - 1)$. Now the problem consists in choosing J, L and n_μ in a way leading to the desired result. In a further reduction step estimates of $x_{n_\mu}(id^*_\mu)$ are traced back to estimates of $x_k(id : \ell^m_{p_0} \to \ell^m_p)$. To estimate from below one has to figure out appropriate subspaces of $s^{t,\Omega}_{p_0,q}f$. Then, also in this case, we can reduce to the estimates of $x_k(id : \ell^m_{p_0} \to \ell^m_p)$. All what is needed about these number have been obtained by Lubitz [65], König [55] and Caetano [13, 14].

It is important to emphasize that this method does not work in the extreme cases given by either $p = 1$ or $p = \infty$. To overcome this obstacle we employ interpolation properties of Weyl numbers and a relation with absolutely $(r, 2)$-summing norms. In this situation we also obtain the right order of convergence of the Weyl numbers. It turns out that the result we obtain for the Weyl numbers in case of $L_\infty(\Omega)$ is the sharp lower estimate for approximation and Gelfand numbers of these embeddings which have been left open since 1960.

Bernstein numbers were first introduced by Mityagin and Pelczynski [73]. In the context of widths, the notion goes back to the work of Tikhomirov [124]. In approximation theory, Bernstein numbers serve as the lower bound for Gelfand, Kolmogorov numbers and nonlinear n-widths, see [22, 83]. Unlike in the case of Weyl numbers, the decomposition technique does not work for Bernstein numbers since they are not additive s-numbers. Concerning the estimate from above we show that Bernstein numbers are dominated by entropy numbers, i.e.,

$$b_n(T) \leq 2\sqrt{2}e_n(T), \quad n \in \mathbb{N},$$

valid for all linear bounded operator T. From this and the inequality

$$b_{2n-1}(T) \leq e\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}, \quad n \in \mathbb{N},$$

in combination with the asymptotic polynomial behaviour of the Weyl numbers we obtain the sharp upper estimate for Bernstein numbers. In particular, we are able to show that

$$\begin{aligned} &b_n(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega)) \\ &\qquad \asymp \min\big\{x_n(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega)),\ e_n(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega))\big\} \end{aligned}$$

in most of the cases, except the case $2 < p_0 < p < \infty$ when $q = p_0$. Our results in this chapter will be compared with those obtained by Galeev [40] in Section 4.5.3.

Beside Weyl numbers, entropy numbers are also a good tool to control eigenvalues of compact operators which is illustrated in the well-known Carl-Triebel inequality

$$|\lambda_n(T)| \leq \sqrt{2}\, e_n(T)\,. \tag{P.2}$$

In Section 4.6 we shall compare our results of Weyl numbers with the already known results of entropy numbers. Based on the those results and the inequalities (P.1), (P.2), we are able to show how Weyl and entropy numbers control the eigenvalues of a compact operator in $L_p(\Omega)$. In some situations, the inequality (P.1) is better than (P.2).

The last section of this chapter is devoted to recall the asymptotic behaviour of some other s-numbers which are closely related to Weyl and Bernstein numbers and play a crucial role in approximation theory and information-based complexity.

This doctoral thesis consists in the following research papers:

1. V.K. Nguyen, Bernstein numbers of embeddings of isotropic and dominating mixed Besov spaces, Math. Nachr. 288 (2015), 1694–1717.

2. V.K. Nguyen, W. Sickel, Weyl numbers of embeddings of tensor product Besov spaces, J. Approx. Theory 200 (2015), 170–220.

3. V.K. Nguyen, Weyl and Bernstein numbers of embeddings of Sobolev spaces with dominating mixed smoothness, J. Complexity 36 (2016), 46–73.

4. V.K. Nguyen, M. Ullrich, T. Ullrich, Change of variable in spaces of mixed smoothness and numerical integration of multivariate functions on the unit cube, Constr. Approx. (2017), DOI: 10.1007/s00365-017-9371-9.

5. V.K. Nguyen, W. Sickel, Isotropic and dominating mixed Besov spaces - a comparison, Functional Analysis, Harmonic Analysis and Image Processing: A collection of papers in honor of Björn Jawerth, Contemp. Math., AMS, to appear.

6. V.K. Nguyen, Gelfand numbers of embeddings of mixed Besov spaces, J. Complexity 41 (2017), 35–57.

7. V.K. Nguyen, W. Sickel, Pointwise multipliers for Sobolev and Besov spaces of dominating mixed smoothness, J. Math. Anal. Appl. 452 (2017), 62–90.

8. V.K. Nguyen, W. Sickel, Isotropic and dominating mixed Lizorkin - Triebel spaces - a comparison, Analysis Mathematica, to appear.

Contents

1 Function spaces of dominating mixed smoothness

1.1 Preliminaries

1.1.1 Notations

As usual, $\mathbb{N}$ denotes the natural numbers, $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\mathbb{Z}$ the integers and $\mathbb{R}$ the real numbers. The letter $\mathbb{C}$ stands for the plane of complex numbers. For a real number a we put $a_+ := \max(a, 0)$ and $[a]$ stands for the integer part of a.

The natural number d is always reserved for the underlying dimension in $\mathbb{R}^d$, $\mathbb{N}^d$, etc, and by $[d]$ we mean $[d] = \{1, \ldots, d\}$. We denote points of the considered Euclidean space by $x, y, z, \ldots$ whose components are numbered from 1 to d, i.e., $x = (x_1, \ldots, x_d)$. If $x, y \in \mathbb{R}^d$, the Euclidean distance is given as usual by $|x|$, $|y|$, and the corresponding Euclidean scalar product is denoted by $x \cdot y$ or $\langle x, y\rangle = \sum_{i=1}^d x_i y_i$. Sometime we use the notation $x \diamond y$ which means $x \diamond y = (x_1 \cdot y_1, \ldots, x_d \cdot y_d)$.

For the multi-index $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ we put

$$|k|_1 := \sum_{i=1}^d k_i \qquad \text{and} \qquad |k|_\infty = \max_{i=1,\ldots,d} k_i .$$

When $k, \ell \in \mathbb{N}_0^d$ we write $k < \ell$ if, and only if, $k_i < \ell_i$ for every $i = 1, \ldots, d$. Similarly, we define the relations $k \le \ell$, $k > \ell$ and $k \ge \ell$. If $a \in \mathbb{N}_0$ by $\bar{a}$ we denote $\bar{a} = (a, \ldots, a) \in \mathbb{N}_0^d$. For $\alpha \in \mathbb{N}_0^d$, the derivative

$$D^\alpha = \frac{\partial^{|\alpha|_1}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}$$

has the usual distributional meaning. Moreover $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$. Sometime we use the notation 2^k for $k = (k_1, \ldots, k_d) \in \mathbb{Z}^d$ which means $2^k = (2^{k_1}, \ldots, 2^{k_d})$. For $e \subset [d]$, $e \neq \emptyset$ we will use the symbol $\mathbb{N}_0^d(e)$ to denote

$$\mathbb{N}_0^d(e) = \{k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d : \ k_i = 0, \ i \notin e\}.$$

Let $L_p(\mathbb{R}^d)$, $0 < p \le \infty$, be the space of all Lebesgue-measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ such that

$$\|f|L_p(\mathbb{R}^d)\| := \Big(\int\limits_{\mathbb{R}^d} |f(x)|^p \, \mathrm{d}x \Big)^{1/p} < \infty$$

with the usual modification if $p = \infty$. For $m \in \mathbb{N}_0$ we denote by $C_0^m(\mathbb{R}^d)$ the collection of all compactly supported functions φ on $\mathbb{R}^d$ which have classical derivatives $D^\alpha \varphi$ uniformly continuous on $\mathbb{R}^d$ for $\alpha \in \mathbb{N}_0^d$ such that $|\alpha|_1 \le m$. Additionally, we define the spaces of infinitely differentiable functions $C^\infty(\mathbb{R}^d)$ and infinitely differentiable functions with compact support $C_0^\infty(\mathbb{R}^d)$. Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. The topological dual of $\mathcal{S}(\mathbb{R}^d)$, the class of tempered distributions, is denoted by $\mathcal{S}'(\mathbb{R}^d)$ (equipped with the weak topology). The Fourier transform on $\mathcal{S}(\mathbb{R}^d)$ is given by

$$\mathcal{F}\varphi(x) = \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} e^{-ixy} \, \varphi(y) \, \mathrm{d}y \,, \qquad x \in \mathbb{R}^d \,.$$

The inverse transformation is denoted by $\mathcal{F}^{-1}$ and defined as $(\mathcal{F}^{-1}\varphi)(x) = \mathcal{F}\varphi(-x)$. As usual, the Fourier transform can be extended to $\mathcal{S}'(\mathbb{R}^d)$ by $(\mathcal{F}f)(\varphi) := f(\mathcal{F}\varphi)$, where $f \in \mathcal{S}'(\mathbb{R}^d)$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$. The mapping $\mathcal{F} : \mathcal{S}'(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ is a bijection.

Let $0 < p, q \le \infty$ and I be an arbitrary countable index set. For a sequence of complex-valued functions $\{f_k\}_{k\in I}$ on $\mathbb{R}^d$, we put

$$\|f_k|\ell_q(L_p)\| = \Big(\sum_{k\in I} \big\|f_k|L_p(\mathbb{R}^d)\big\|^q\Big)^{1/q} = \Bigg(\sum_{k\in I}\Big(\int\limits_{\mathbb{R}^d} |f_k(x)|^p \,\mathrm{d}x\Big)^{q/p}\Bigg)^{1/q}$$

and

$$\|f_k|L_p(\ell_q)\| = \Big\|\Big(\sum_{k\in I}|f_k|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| = \Bigg(\int\limits_{\mathbb{R}^d}\Big(\sum_{k\in I}|f_k(x)|^q\Big)^{p/q}\mathrm{d}x\Bigg)^{1/p}$$

with usual modification if $\max(p, q) = \infty$.

If X and Y are two quasi-Banach spaces, then $\mathcal{L}(X, Y)$ denotes the space of all continuous linear operators from X into Y. The (quasi-)norm of an element x in X will be denoted by $\|x|X\|$. The symbol $X \hookrightarrow Y$ indicates that the embedding is continuous. The notation X' stands for the topological dual space of X. As usual, the symbols $c, c_1, \ldots, C, C_1, \ldots$ denote positive constants which depend only on the fixed parameters t, p, q and probably on auxiliary functions, unless otherwise stated. Sometimes we will use the symbols "$\lesssim$" and "$\gtrsim$" instead of "$\le$" and "$\ge$", respectively. The meaning of $A \lesssim B$ is given by: there exists a constant $c > 0$ such that $A \le c\,B$. Similarly $\gtrsim$ is defined. The notation $A \asymp B$ will be used as an abbreviation of $A \lesssim B \lesssim A$. For a finite set G the symbol $|G|$ denotes the cardinality of this set. Finally, the symbols id will be used for identity operators. The notation $id^m_{p_0,p}$ refers to the identity

$$id^m_{p_0,p} :\ \ell^m_{p_0} \to \ell^m_p\,.$$

1.1.2 Maximal operators and Fourier multipliers

In this section we will collect some maximal inequalities and Fourier multiplier assertions for scalar and vector-valued L_p-spaces which play an important role in our investigations. The constructions are based on the Hardy-Littlewood maximal operator and the maximal operator of Peetre. For a locally integrable function $f : \mathbb{R}^d \to \mathbb{C}$ we denote by $\mathcal{M}f(x)$ the Hardy-Littlewood maximal function defined as

$$(\mathcal{M}f)(x) = \sup_{x\in Q}\frac{1}{|Q|}\int\limits_Q |f(y)|\,\mathrm{d}y\,, \qquad x \in \mathbb{R}^d\,, \tag{1.1}$$

where the supremum is taken over all cubes with sides parallel to the coordinate axes containing x. A vector valued generalization of the classical Hardy-Littlewood maximal inequality is due to Fefferman and Stein [35].

Theorem 1.1. *For $1 < p < \infty$ and $1 < q \le \infty$ there exists a constant $C > 0$, such that*

$$\|\mathcal{M}f_k|L_p(\ell_q)\| \le C\|f_k|L_p(\ell_q)\|$$

holds for all sequences $\{f_k\}_{k\in\mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

We require a direction-wise version of (1.1)

$$(\mathcal{M}_i f)(x) = \sup_{s>0} \frac{1}{2s} \int\limits_{x_i-s}^{x_i+s} |f(x_1,\ldots,x_{i-1},\xi,x_{i+1},\ldots,x_d)|\,\mathrm{d}\xi, \qquad i=1,\ldots,d\,.$$

For $e \subset [d]$, $e \neq \emptyset$, we denote $\mathcal{M}_e = \prod_{i\in e} \mathcal{M}_i$ where $(\mathcal{M}_j\mathcal{M}_\ell f)(x) = \mathcal{M}_j(\mathcal{M}_\ell f)(x)$. The following version of the Fefferman-Stein maximal inequality is due Stöckert [110], see also Bagby [2].

Theorem 1.2. *Let $1 < p < \infty$ and $1 < q \leq \infty$. Then there exists a constant $C > 0$ such that for any $i = 1,\ldots,d$*

$$\|\mathcal{M}_i f_k|L_p(\ell_q)\| \leq C\|f_k|L_p(\ell_q)\|$$

holds for all sequences $\{f_k\}_{k\in\mathbb{N}_0^d}$ of locally Lebesgue-integrable functions on $\mathbb{R}^d$.

Remark 1.3. Iteration of Theorem 1.2 yields a similar boundedness property for every operator $\mathcal{M}_e$, $e \subset [d]$. Note that with $p = 1$ and/or $q = 1$ the statements in Theorems 1.1 and 1.2 become false, see, e.g., Stein [109, Section 2.5].

As a consequence of Theorem 1.2 we have the following result. In this situation we can extend to the case $q = 1$, see Remark 1.3. We follow essentially the proof for isotropic setting by Yamazaki [142] .

Proposition 1.4. *Let $1 < p < \infty$ and $1 \leq q \leq \infty$. Suppose $\phi \in \mathcal{S}(\mathbb{R}^d)$. Then there exists a constant $C > 0$ such that*

$$\big\|\mathcal{F}^{-1}[\phi(2^{-k}\diamond y)\mathcal{F}f_k(y)](\cdot)\big|L_p(\ell_q)\big\| \leq C\|f_k\big|L_p(\ell_q)\|$$

holds for all sequences $\{f_k\}_{k\in\mathbb{N}_0^d} \in L_p(\ell_q)$.

Proof. *Step 1.* The case $1 < q \leq \infty$. Observe that for $k \in \mathbb{N}_0^d$ we have

$$\mathcal{F}^{-1}[\phi(2^{-k}\diamond\cdot)\mathcal{F}f_k(\cdot)](x) = (2\pi)^{-\frac{d}{2}}2^{|k|_1}\int\limits_{\mathbb{R}^d}(\mathcal{F}^{-1}\phi)(2^k\diamond y)f_k(x-y)\,\mathrm{d}y\,. \tag{1.2}$$

Let $\alpha > 1$. The assumption $\phi \in \mathcal{S}(\mathbb{R}^d)$ implies

$$\begin{aligned}
&\Big|\int\limits_{\mathbb{R}^d}(\mathcal{F}^{-1}\phi)(2^k\diamond y)f_k(x-y)\,\mathrm{d}y\Big| \\
&\quad\leq \sup_{y\in\mathbb{R}^d}\Big\{\Big(\prod_{i=1}^d(1+|2^{k_i}y_i|^2)^{\frac{\alpha}{2}}\Big)|(\mathcal{F}^{-1}\phi)(2^k\diamond y)|\Big\}\int\limits_{\mathbb{R}^d}\Big(\prod_{i=1}^d(1+|2^{k_i}y_i|^2)^{-\frac{\alpha}{2}}\Big)|f_k(x-y)|\,\mathrm{d}y \\
&\quad\leq c\int\limits_{\mathbb{R}^d}\Big(\prod_{i=1}^d(1+|2^{k_i}y_i|^2)^{-\frac{\alpha}{2}}\Big)|f_k(x-y)|\,\mathrm{d}y
\end{aligned} \tag{1.3}$$

with a constant c independent of k and f_k. For $\ell \in \mathbb{Z}^d$ we put

$$P(k,\ell) = \big\{x\in\mathbb{R}^d:\ 2^{-k_i}2^{\ell_i} \leq |x_i| < 2^{-k_i}2^{\ell_i+1},\quad i=1,\ldots,d\big\}.$$

Then we obtain from (1.3)

$$\begin{aligned} &\Big|\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\phi)(2^k\diamond y)f_k(x-y)\,\mathrm{d}y\Big| \\ &\qquad\qquad \le c\sum_{\ell\in\mathbb{Z}^d}\Big(\sup_{y\in P(k,\ell)}\prod_{i=1}^d(1+|2^{k_i}y_i|^2)^{-\frac{\alpha}{2}}\Big)\int_{P(k,\ell)}|f_k(x-y)|\,\mathrm{d}y. \end{aligned} \tag{1.4}$$

Applying $\mathcal{M}_{[d]}$ to the integral on the right-hand side of (1.4) yields

$$\begin{aligned} \Big|\int_{\mathbb{R}^d}(\mathcal{F}^{-1}\phi)(2^k\diamond y)f_k(x-y)\,\mathrm{d}y\Big| &\le c(\mathcal{M}_{[d]}f_k)(x)\sum_{\ell\in\mathbb{Z}^d}2^{-|k|_1}\sup_{y\in P(k,\ell)}\prod_{i=1}^d\frac{2^{\ell_i}}{(1+|2^{k_i}y_i|^2)^{\frac{\alpha}{2}}} \\ &\le c_1\,2^{-|k|_1}(\mathcal{M}_{[d]}f_k)(x)\sum_{\ell\in\mathbb{Z}^d}\prod_{i=1}^d\frac{2^{\ell_i}}{(1+2^{\ell_i})^{\alpha}} \\ &\le c_2\,2^{-|k|_1}(\mathcal{M}_{[d]}f_k)(x)\,. \end{aligned}$$

Inserting this into (1.2) we arrive at

$$\mathcal{F}^{-1}[\phi(2^{-k}\diamond\cdot)\mathcal{F}f_k(\cdot)](x)\le c_3(\mathcal{M}_{[d]}f_k)(x)\,.$$

Now the desired estimate follows from Theorem 1.2.
Step 2. The case $q=1$. From Step 1 we derive that the linear operator

$$T:\big\{f_k\big\}_k\to\big\{\mathcal{F}^{-1}[\phi(2^{-k}\diamond y)\mathcal{F}f_k(y)]\big\}_k$$

is bounded from $L_{p'}(\ell_\infty)$ into itself. By a duality argument we conclude that the dual operator T' of T is bounded from $[L_{p'}(\ell_\infty)]'$ into itself. That is

$$\big\|\mathcal{F}^{-1}[\phi(2^{-k}\diamond y)\mathcal{F}f_k(y)](\cdot)\,\big|\,[L_{p'}(\ell_\infty)]'\big\|\le C\|\,f_k\,|\,[L_{p'}(\ell_\infty)]'\|$$

for all $\{f_k\}_{k\in\mathbb{N}_0^d}\in[L_{p'}(\ell_\infty)]'$. Here without loss of generality we have assumed that ϕ is the even function. Since $[L_{p'}(\ell_\infty)]'$ is double dual space of $L_p(\ell_1)$ then $L_p(\ell_1)$ can be canonically identified with a closed subspace of $[L_{p'}(\ell_\infty)]'$. We have

$$\big\|\mathcal{F}^{-1}[\phi(2^{-k}\diamond y)\mathcal{F}f_k(y)](\cdot)\big|[L_{p'}(\ell_\infty)]'\big\|\le C\|f_k\big|L_p(\ell_1)\|$$

for all $\{f_k\}_{k\in\mathbb{N}_0^d}\in L_p(\ell_1)$. We put

$$\begin{aligned} \mathcal{A}=\Big\{&\{u_k\}_{k\in\mathbb{N}_0^d},\ u_k\in\mathcal{S}(\mathbb{R}^d)\ \text{ and} \\ &\qquad u_k\equiv 0\ \text{ for all but a finite number of }\ k\Big\}. \end{aligned}$$

It is obvious that

$$\big\{\mathcal{F}^{-1}[\phi(2^{-k}\diamond y)\mathcal{F}u_k(y)]\big\}_{k\in\mathbb{N}_0^d}\in\mathcal{A}\subset L_p(\ell_1)$$

if $\{u_k\}_{k\in\mathbb{N}_0^d}\in\mathcal{A}$. Because $\mathcal{A}$ is dense in $L_p(\ell_1)$ we conclude that

$$\big\|\mathcal{F}^{-1}[\phi(2^{-k}\diamond y)\mathcal{F}f_k(y)](\cdot)\big|L_p(\ell_1)\big\|\le C\|f_k\big|L_p(\ell_1)\|$$

holds true for all $\{f_k\}_{k\in\mathbb{N}_0^d}\in L_p(\ell_1)$. The proof is complete. ■

We proceed by considering spaces of entire analytic functions. For $0 < p \le \infty$ and a compact subset $\Omega \subset \mathbb{R}^d$ we introduce the notation

$$L_p^\Omega(\mathbb{R}^d) = \left\{ f \in \mathcal{S}'(\mathbb{R}^d) : \ \operatorname{supp} \mathcal{F}f \subset \Omega \,, \ f \in L_p(\mathbb{R}^d) \right\} .$$

The following adapted version of the famous Nikol'skij inequality is due to B. Stöckert [110] and A. P. Uninskij [139], see also [104, Theorem 1.6.2].

Theorem 1.5. *Let $0 < p_0 \le p \le \infty$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Let $\Omega = [-b_1, b_1] \times \ldots \times [-b_d, b_d]$, $b_i > 0$, $i = 1, \ldots, d$. Then there exists a positive constant C, independent of $(b_1, \ldots, b_d)$, such that*

$$\|D^\alpha f | L_p(\mathbb{R}^d)\| \le C \Big(\prod_{i=1}^d b_i^{\alpha_i + \frac{1}{p_0} - \frac{1}{p}} \Big) \|f | L_{p_0}(\mathbb{R}^d)\|$$

holds for all $f \in L_{p_0}^\Omega(\mathbb{R}^d)$.

Let us next recall a Fourier multiplier assertion for $L_p^\Omega(\mathbb{R}^d)$, see [130, Proposition 1.5.1].

Lemma 1.6. *Let Ω and Γ be compact subsets of $\mathbb{R}^d$. Let further $0 < p \le \infty$ and $u := \min(p, 1)$. Then there exists a positive constant C such that*

$$\|\mathcal{F}^{-1} M \mathcal{F} f | L_p(\mathbb{R}^d)\| \le C \, \|\mathcal{F}^{-1} M | L_u(\mathbb{R}^d)\| \cdot \|f | L_p(\mathbb{R}^d)\|$$

holds for all $f \in L_p^\Omega(\mathbb{R}^d)$ and all $\mathcal{F}^{-1} M \in L_u^\Gamma(\mathbb{R}^d)$.

The following construction of a maximal function is essentially due to Peetre, but based on earlier work of Fefferman and Stein. Let $a > 0$ and $b = (b_1, \ldots, b_d)$, $b_i > 0$, $i = 1, \ldots, d$ be fixed. Let f be a regular distribution such that $\mathcal{F}f$ is compactly supported. We define the Peetre maximal function $P_{b,a} f$ by

$$P_{b,a} f(x) = \sup_{z \in \mathbb{R}^d} \frac{|f(x-z)|}{\prod_{i=1}^d (1 + |b_i z_i|)^a} = \sup_{y \in \mathbb{R}^d} \frac{|f(y)|}{\prod_{i=1}^d (1 + |b_i (x_i - y_i)|)^a}.$$

Lemma 1.7. *Let $0 < p \le \infty$ and $\Omega \subset \mathbb{R}^d$ be a compact set. Let further $a > 0$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}_0^d$. Then there exist two constants $C_1, C_2 > 0$ (independent of f) such that*

$$P_{(1,\ldots,1),a}(D^\alpha f)(x) \,\le\, C_1 P_{(1,\ldots,1),a} f(x) \,\le\, C_2 \big(\mathcal{M}(|f|^{1/a})\big)^a(x)$$

holds for all $f \in L_p^\Omega(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.

Remark 1.8. The proof of Lemma 1.7 can be found in [104, Theorem 1.6.4]. Note that the constants C_1, C_2 there depend on Ω. Let $\Omega = [-b_1, b_1] \times \ldots \times [-b_d, b_d]$, $b_i > 0$, $i = 1, \ldots, d$ and $\operatorname{supp}(\mathcal{F}f) \subset \Omega$. Then applying Lemma 1.7 for the function $f(x_1/b_1, \ldots, x_d/b_d)$ we obtain

$$P_{b,a}(D^\alpha f)(x) \,\le\, C_1 \Big(\prod_{i=1}^d b_i^{\alpha_i} \Big) P_{b,a} f(x) \,\le\, C_2 \Big(\prod_{i=1}^d b_i^{\alpha_i} \Big) \big(\mathcal{M}(|f|^{1/a})\big)^a(x)$$

for all $f \in L_p^\Omega(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. The constants C_1, C_2 now are independent of f and $b = (b_1, \ldots, b_d)$.

Let us introduce vector-valued L_p-spaces of entire analytic functions.

Definition 1.9. Let $0 < p, q \leq \infty$. Let $\Omega = \{\Omega_k\}_{k\in\mathbb{N}_0^d}$ be a sequence of compact subsets in $\mathbb{R}^d$. Then we define

$$L_p^\Omega(\ell_q) = \Big\{\{f_k\}_{k\in\mathbb{N}_0^d}:\ f_k \in \mathcal{S}'(\mathbb{R}^d),\ \mathrm{supp}\,(\mathcal{F}f_k) \subset \Omega_k \text{ if } \ k \in \mathbb{N}_0^d,\ \|f_k|L_p(\ell_q)\| < \infty\Big\}.$$

We will need the vector-valued Peetre maximal inequality which is a direct consequence of Lemma 1.7 together with Theorem 1.2. For more details see [104, Theorem 1.10.2] and [45, Proposition 2.3.4].

Theorem 1.10. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\Omega = \{\Omega_k\}_{k\in\mathbb{N}_0^d}$ be a sequence of compact subsets of $\mathbb{R}^d$*

$$\Omega_k = \big\{x \in \mathbb{R}^d:\ |x_{k_i}| \leq b_{k_i},\ i = 1, \ldots, d\big\}, \quad k \in \mathbb{N}_0^d, \tag{1.5}$$

with $b_k = (b_{k_1}, \ldots, b_{k_d}) \in \mathbb{R}^d$, $b_{k_i} > 0$, $i = 1, \ldots, d$. Let further $a > \frac{1}{\min(p,q)}$. Then there exists a positive constant C independent of $\{b_k\}_{k\in\mathbb{N}_0^d}$ such that

$$\big\|P_{b_k,a}f_k\big|L_p(\ell_q)\big\| \leq C\|f_k|L_p(\ell_q)\|$$

holds for all systems $\{f_k\}_{k\in\mathbb{N}_0^d} \in L_p^\Omega(\ell_q)$.

Finally, we recall a Fourier multiplier assertion for the spaces $L_p^\Omega(\ell_q)$. We refer to [104, Theorem 1.10.3], see also [45, Proposition 2.3.5].

Lemma 1.11. *Let $0 < p < \infty$, $0 < q \leq \infty$ and $\Omega = \{\Omega_k\}_{k\in\mathbb{N}_0^d}$ be the sequence of compact subsets given in* (1.5)*. Let $r > \frac{1}{\min(p,q)} + \frac{1}{2}$. Then there exists a constant C such that*

$$\|\mathcal{F}^{-1}M_k\mathcal{F}f_k \,|\, L_p(\ell_q)\| \ \leq\ C \sup_{\ell\in\mathbb{N}_0^d} \|M_\ell(b_\ell \diamond \cdot)|S_2^rH(\mathbb{R}^d)\| \cdot \|f_k|L_p(\ell_q)\|$$

holds for all systems $\{f_k\}_{k\in\mathbb{N}_0^d} \in L_p^\Omega(\ell_q)$ and all systems $\{M_k\}_{k\in\mathbb{N}_0^d} \in S_2^rH(\mathbb{R}^d)$. For a definition of the space $S_2^rH(\mathbb{R}^d)$, see Definition 1.19.

1.1.3 Isotropic Besov and Triebel-Lizorkin spaces

We begin this section with the definition of isotropic Sobolev spaces and their counterparts of fractional order.

Definition 1.12. Let $1 < p < \infty$.
(i) Let $m \in \mathbb{N}_0$. We define the Sobolev space

$$W_p^m(\mathbb{R}^d) = \Big\{f \in L_p(\mathbb{R}^d) : \|f|W_p^m(\mathbb{R}^d)\| = \sum_{\alpha\in\mathbb{N}_0^d,\,|\alpha|_1\leq m} \|D^\alpha f|L_p(\mathbb{R}^d)\| < \infty\Big\}. \tag{1.6}$$

(ii) Let $t \in \mathbb{R}$. Then the space $H_p^t(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|H_p^t(\mathbb{R}^d)\| = \big\|\mathcal{F}^{-1}\big[(1 + |y|^2)^{t/2}\mathcal{F}f\big](\cdot)\big|L_p(\mathbb{R}^d)\big\|$$

is finite. Here $y \in \mathbb{R}^d$.

Remark 1.13. The derivatives in (1.6) have to be understood in the sense of distributions. The space $H^t_p(\mathbb{R}^d)$ is also called Bessel-potential space. It is obvious that $W^0_p(\mathbb{R}^d) = H^0_p(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. In general, if $m \in \mathbb{N}_0$ we have $W^m_p(\mathbb{R}^d) = H^m_p(\mathbb{R}^d)$ in the sense of equivalent norms, see [130, Theorem 2.5.6].

We now turn to isotropic Besov and Triebel-Lizorkin spaces.

Definition 1.14. Let $\Phi(\mathbb{R}^d)$ be the collection of all systems $\{\phi_j\}_{j=0}^\infty \in \mathcal{S}(\mathbb{R}^d)$ such that

$$\begin{cases} \operatorname{supp} \phi_0 \subset \{x \in \mathbb{R}^d : |x| \le 2\} \\ \operatorname{supp} \phi_j \subset \{x \in \mathbb{R}^d : 2^{j-1} \le |x| \le 2^{j+1}\} \quad \text{if} \quad j \in \mathbb{N}; \end{cases}$$

for every multi-index $\alpha \in \mathbb{N}^d_0$ there exists a positive constants C_α such that

$$2^{j|\alpha|_1} |D^\alpha \phi_j(x)| \le C_\alpha \quad \text{for all} \quad j \in \mathbb{N}_0 \quad \text{and } x \in \mathbb{R}^d$$

and

$$\sum_{j=0}^\infty \phi_j(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

Remark 1.15. We shall call $\{\phi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^d)$ a smooth dyadic decomposition of unity on $\mathbb{R}^d$. The class $\Phi(\mathbb{R}^d)$ is not empty. Let $\phi_0 \in C_0^\infty(\mathbb{R}^d)$ be a non-negative function such that $\phi_0(x) = 1$ if $|x| \le 1$ and $\phi_0(x) = 0$ if $|x| \ge \frac{3}{2}$. For $j \in \mathbb{N}$ we define

$$\phi_j(x) := \phi_0(2^{-j}x) - \phi_0(2^{-j+1}x), \qquad x \in \mathbb{R}^d\,.$$

Then it is not difficult to verify that this system belongs to the class $\Phi(\mathbb{R}^d)$.

Definition 1.16. Let $0 < p, q \le \infty$, $t \in \mathbb{R}$ and let $\{\phi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^d)$.
(i) The Besov space $B^t_{p,q}(\mathbb{R}^d)$ is then the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \,| B^t_{p,q}(\mathbb{R}^d)\|^\phi := \Big(\sum_{j=0}^\infty 2^{jtq} \, \| \mathcal{F}^{-1}(\phi_j \mathcal{F} f) \,| L_p(\mathbb{R}^d)\|^q \Big)^{1/q}$$

is finite.
(ii) Let $p < \infty$. The Lizorkin-Triebel space $F^t_{p,q}(\mathbb{R}^d)$ is then the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \,| F^t_{p,q}(\mathbb{R}^d)\|^\phi := \Big\| \Big(\sum_{j=0}^\infty 2^{jtq} \, | \mathcal{F}^{-1}(\phi_j \mathcal{F} f)(\cdot) \,|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\|$$

is finite.

Remark 1.17. As usual, the symbol $A^t_{p,q}(\mathbb{R}^d)$ stands for $B^t_{p,q}(\mathbb{R}^d)$ and $F^t_{p,q}(\mathbb{R}^d)$ respectively. We call these spaces isotropic because they are invariant under rotations. The isotropic Besov and Triebel-Lizorkin spaces are quasi-Banach spaces (Banach spaces if $\min(p, q) \ge 1$) and do not depend on the system $\{\phi_j\}_{j=0}^\infty \in \Phi(\mathbb{R}^d)$ in the sense of equivalent quasi-norms. The two scales $B^t_{p,q}(\mathbb{R}^d)$ and $F^t_{p,q}(\mathbb{R}^d)$ are closely related and $B^{t_1}_{p_1,q_1}(\mathbb{R}^d) = F^{t_2}_{p_2,q_2}(\mathbb{R}^d)$ if and only if $t_1 = t_2$, $p_1 = p_2 = q_1 = q_2$.

Remark 1.18. Isotropic Besov and Triebel-Lizorkin spaces are discussed in various monographs. Let us refer to Bergh, Löfström [11], Nikol'skij [78], Peetre [82], and Triebel [130, 131, 132]. They cover many classical scales of function spaces such as Sobolev spaces, local Hardy spaces, or Hölder-Zygmund spaces, see [130, Chapter 2]. For more properties we refer to the above-mentioned monographs.

1.2 Function spaces of dominating mixed smoothness

1.2.1 Besov and Triebel-Lizorkin spaces of dominating mixed smoothness

In this section we shall define the function spaces of dominating mixed smoothness on $\mathbb{R}^d$ and recall their basic properties. First, let us introduce the Sobolev and Bessel-potential spaces of dominating mixed smoothness.

Definition 1.19. Let $1 < p < \infty$.
(i) Let $m \in \mathbb{N}_0$. Then the Sobolev space of dominating mixed smoothness $S_p^m W(\mathbb{R}^d)$ is the collection of all $f \in L_p(\mathbb{R}^d)$ such that all distributional derivatives $D^\alpha f$ with $|\alpha|_\infty \le m$ belong to $L_p(\mathbb{R}^d)$. We put

$$\| f \,| S_p^m W(\mathbb{R}^d)\| := \sum_{|\alpha|_\infty \le m} \| D^\alpha f \,| L_p(\mathbb{R}^d)\| \,.$$

(ii) Let $t \in \mathbb{R}$. Then Bessel-potential space of dominating mixed smoothness $S_p^t H(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f|S_p^t H(\mathbb{R}^d)\| = \big\|\mathcal{F}^{-1}\big[(1+y_1^2)^{t/2} \cdot \ldots \cdot (1+y_d^2)^{t/2} \mathcal{F} f\big](\cdot)\big| L_p(\mathbb{R}^d)\big\|$$

is finite.

Remark 1.20. In the literature sometimes the notation $MW_p^t(\mathbb{R}^d)$ is used instead of $S_p^t H(\mathbb{R}^d)$. From the Definition 1.19 we have at once $S_p^0 W(\mathbb{R}^d) = S_p^0 H(\mathbb{R}^d) = L_p(\mathbb{R}^d)$. In general, if $m \in \mathbb{N}_0$ we have $S_p^m W(\mathbb{R}^d) = S_p^m H(\mathbb{R}^d)$ in the sense of equivalent norms, see [104, Theorem 2.3.1].

Remark 1.21. The scale $S_p^t H(\mathbb{R}^d)$, of course also $S_p^m W(\mathbb{R}^d)$, has the cross-norm property, i.e., if $f_i \in H_p^t(\mathbb{R})$, $i = 1, \ldots, d$, then $f = f_1 \otimes \ldots \otimes f_d \in S_p^t H(\mathbb{R}^d)$ and

$$\| f \,| S_p^t H(\mathbb{R}^d)\| = \prod_{i=1}^d \| f_i \,| H_p^t(\mathbb{R})\| \,.$$

Let $\{\varphi_j\}_{j=0}^\infty \in \Phi(\mathbb{R})$ be a smooth dyadic decomposition of unity on $\mathbb{R}$, see Definition 1.14. For $k = (k_1, \ldots, k_d) \in \mathbb{N}_0^d$ the function $\varphi_k \in C_0^\infty(\mathbb{R}^d)$ is defined as a tensor product, i.e.,

$$\varphi_k(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d)\,, \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Then we have

$$\sum_{k \in \mathbb{N}_0^d} \varphi_k(x) = 1 \qquad \text{for all} \quad x \in \mathbb{R}^d\,.$$

Again the system $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ is a smooth dyadic decomposition of unity on $\mathbb{R}^d$. Using this kind of notation, we are ready to give the definition of the spaces $S_{p,q}^t B(\mathbb{R}^d)$ and $S_{p,q}^t F(\mathbb{R}^d)$ in the Fourier analytic approach.

Definition 1.22. Let $t \in \mathbb{R}$, $0 < p, q \le \infty$ and let $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ be the above system.
(i) The Besov space of dominating mixed smoothness $S_{p,q}^t B(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \,| S_{p,q}^t B(\mathbb{R}^d)\|^\varphi := \Big(\sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1 q} \, \| \mathcal{F}^{-1}(\varphi_k \, \mathcal{F} f) \,| L_p(\mathbb{R}^d)\|^q\Big)^{1/q} \tag{1.7}$$

is finite.
(ii) Let $0 < p < \infty$. The Lizorkin-Triebel space of dominating mixed smoothness $S^t_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\| f \,|S^t_{p,q}F(\mathbb{R}^d)\|^{\varphi} := \Big\| \Big(\sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1 q} \,|\, \mathcal{F}^{-1}(\varphi_k\, \mathcal{F} f)\,|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \tag{1.8}$$

is finite.

Remark 1.23. We will use the notation $S^t_{p,q}A(\mathbb{R}^d)$ to refer to both scales of function spaces. From the above definition we have $S^t_{p,p}B(\mathbb{R}^d) = S^t_{p,p}F(\mathbb{R}^d)$ for $t \in \mathbb{R}$ and $0 < p < \infty$. If $d = 1$ we obtain $S^t_{p,q}A(\mathbb{R}) = A^t_{p,q}(\mathbb{R})$. Besov and Lizorkin-Triebel spaces of dominating mixed smoothness also have cross-quasi-norms in the sense that if $f_i \in A^t_{p,q}(\mathbb{R})$ for $i = 1, \ldots, d$ then we have $f = f_1 \otimes \ldots \otimes f_d \in S^t_{p,q}A(\mathbb{R}^d)$ and

$$\| f \,|S^t_{p,q}A(\mathbb{R}^d)\| = \prod_{i=1}^{d} \| f_i \,|A^t_{p,q}(\mathbb{R})\| \,.$$

Remark 1.24. Both scales of function spaces are quasi-Banach spaces (Banach spaces if $\min(p, q) \geq 1$). The spaces $S^t_{p,q}A(\mathbb{R}^d)$ are independent of the chosen decomposition of unity up to equivalence of the quasi-norms in the corresponding spaces, see e.g. [104, Theorem 2.2.4]. Hence, we shall not indicate the index "φ" in the sequel, i.e., we simply write $\|f|S^t_{p,q}A(\mathbb{R}^d)\|$ instead of $\|f|S^t_{p,q}A(\mathbb{R}^d)\|^{\varphi}$. For this reason, when working with these spaces we shall use the following special decomposition of unity $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ which is helpful in proofs. Let $\varphi_0 \in C_0^\infty(\mathbb{R})$ be a non-negative even function with $\varphi_0(\xi) = 1$ if $\xi \in [-1, 1]$ and $\operatorname{supp} \varphi_0 \subset [-\frac{3}{2}, \frac{3}{2}]$. For $j \in \mathbb{N}$ we define

$$\varphi_j(\xi) = \varphi_0(2^{-j}\xi) - \varphi_0(2^{-j+1}\xi), \qquad \xi \in \mathbb{R}\,,$$

and $\varphi_k(x) := \varphi_{k_1}(x_1) \cdot \ldots \cdot \varphi_{k_d}(x_d)$ for $k \in \mathbb{N}_0^d$, $x \in \mathbb{R}^d$.

Remark 1.25. As in the case of isotropic spaces, the counterpart of Definition 1.22 (ii) for $p = \infty$ does not make sense since the corresponding spaces depend on the choice of the system $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$. In this thesis, when we write $S^t_{\infty,\infty}F(\mathbb{R}^d)$ we mean $S^t_{\infty,\infty}F(\mathbb{R}^d) := S^t_{\infty,\infty}B(\mathbb{R}^d)$.

Remark 1.26. These scales $S^t_{p,q}A(\mathbb{R}^d)$ cover many classical spaces as special cases. The spaces $S^t_{p,q}B(\mathbb{R}^d)$ contain the classical spaces of S.M. Nikol'skij $S^t_{p,\infty}B(\mathbb{R}^d)$. In particular the class $S^t_{\infty,\infty}B(\mathbb{R}^d)$ coincides with a version of Höder-Zygmund class $\mathcal{Z}^t_{\text{mix}}(\mathbb{R}^d)$, see Remark 1.56. The scales of functions $S^t_{p,q}F(\mathbb{R}^d)$ are generalizations of $S^t_pH(\mathbb{R}^d)$. This is a consequence of the Littlewood-Paley type assertion for spaces of dominating mixed smoothness which will be stated in the following.

Theorem 1.27. *Let $t \in \mathbb{R}$ and $1 < p < \infty$. Then we have $S^t_{p,2}F(\mathbb{R}^d) = S^t_pH(\mathbb{R}^d)$ in the sense of equivalent norms, i.e, it holds*

$$\| f \,|S^t_pH(\mathbb{R}^d)\| \asymp \Big\| \Big(\sum_{k\in\mathbb{N}_0^d} 2^{2t|k|_1} |\mathcal{F}^{-1}(\varphi_k \mathcal{F} f)|^2 \Big)^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| \tag{1.9}$$

for all $f \in S^t_pH(\mathbb{R}^d)$. For $t = 0$ we get back the Littlewood-Paley assertion $L_p(\mathbb{R}^d) = S^0_{p,2}F(\mathbb{R}^d)$.

Remark 1.28. Let χ_0 be the characteristic function of $(-1,1)$. Let further χ_j, $j \in \mathbb{N}$, be the characteristic function of $(-2^j, -2^{j-1}] \cup [2^{j-1}, 2^j)$. For $k \in \mathbb{N}_0^d$ we define $\chi_k(x)$, $x \in \mathbb{R}^d$, as a tensor product

$$\chi_k(x) = \chi_{k_1}(x_1) \cdot \ldots \cdot \chi_{k_d}(x_d) \,. \tag{1.10}$$

If we replace the system $\{\varphi_k\}_k$ by $\{\chi_k\}_k$ in (1.9) we then obtain an equivalent norm in $S^t_p H(\mathbb{R}^d)$, $1 < p < \infty$.

Remark 1.29. The proof of $L_p(\mathbb{R}^d) = S^0_{p,2}F(\mathbb{R}^d)$ can be found in Lizorkin [63, 64] or Nikol'skij [78, Section 1.5.6]. To prove (1.9) one has to use $L_p(\mathbb{R}^d) = S^0_{p,2}F(\mathbb{R}^d)$ in connection with the lifting property of function spaces of dominating mixed smoothness, see Theorem 1.31. We refer to [104, Section 2.3] for more details.

Definition 1.30. Let $\rho \in \mathbb{R}$. Then we define the lifting operator by

$$I_\rho f := \mathcal{F}^{-1}\big[(1+y_1^2)^{\rho/2} \cdot \ldots \cdot (1+y_d^2)^{\rho/2}\mathcal{F}f\big]\,, \qquad f \in \mathcal{S}'(\mathbb{R}^d).$$

Theorem 1.31. *Let $0 < p,q \le \infty$ and $t, \rho \in \mathbb{R}$.*
(i) *Then I_ρ maps $S^t_{p,q}B(\mathbb{R}^d)$ isomorphically onto $S^{t-\rho}_{p,q}B(\mathbb{R}^d)$ and $\|I_\rho f|S^{t-\rho}_{p,q}B(\mathbb{R}^d)\|$ is an equivalent quasi-norm in $S^t_{p,q}B(\mathbb{R}^d)$.*
(ii) *Let $0 < p < \infty$. Then I_ρ maps $S^t_{p,q}F(\mathbb{R}^d)$ isomorphically onto $S^{t-\rho}_{p,q}F(\mathbb{R}^d)$ and $\|I_\rho f|S^{t-\rho}_{p,q}F(\mathbb{R}^d)\|$ is an equivalent quasi-norm in $S^t_{p,q}F(\mathbb{R}^d)$.*

The proof of Theorem 1.31 may be found in [104, Section 2.3]. Let us now recall some basic continuous embeddings of functions spaces of dominating mixed smoothness. We refer to the monograph [104, Chapter 2] and Hansen, Vybiral [47].

Lemma 1.32. (i) *Let $t \in \mathbb{R}$ and $0 < p,q,u \le \infty$ (with $p < \infty$ in the F-case). If $\varepsilon > 0$ then we have*

$$S^{t+\varepsilon}_{p,q}A(\mathbb{R}^d) \hookrightarrow S^t_{p,u}A(\mathbb{R}^d)\,.$$

(ii) *Let $t \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \le \infty$. Then we have*

$$S^t_{p,\min(p,q)}B(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow S^t_{p,\max(p,q)}B(\mathbb{R}^d)\,.$$

Lemma 1.33. *Let $t \in \mathbb{R}$ and $0 < p,q \le \infty$ (with $p < \infty$ in the F-case).*
(i) *If $t > \left(\frac{1}{p} - 1\right)_+$ then we have*

$$S^t_{p,q}A(\mathbb{R}^d) \hookrightarrow L_{\max\{p,1\}}(\mathbb{R}^d)\,.$$

(ii) *The embedding*

$$S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$$

holds if and only if either $t > \frac{1}{p}$ or $t = \frac{1}{p}$ and $q \le 1$.
(iii) *The embedding*

$$S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$$

holds if and only if either $t > \frac{1}{p}$ or $t = \frac{1}{p}$ and $p \le 1$.

Lemma 1.34. *Let* $0 < p_0 < p < p_1 \le \infty$ *and* $t_0 - \frac{1}{p_0} = t - \frac{1}{p} = t_1 - \frac{1}{p_1}$. *Then we have*

$$S^{t_0}_{p_0,q_0}B(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow S^{t_1}_{p_1,q_1}B(\mathbb{R}^d)$$

if and only if $0 < q_0 \le p \le q_1 \le \infty$.

Remark 1.35. We recall that $C(\mathbb{R}^d)$ is the space of all complex-valued uniformly continuous bounded functions on $\mathbb{R}^d$, equipped with the norm

$$\|f|C(\mathbb{R}^d)\| := \sup_{x\in\mathbb{R}^d} |f(x)|.$$

1.2.2 Tensor products of Sobolev and Besov spaces

As already mentioned in the last section, function spaces of dominating mixed smoothness have a cross-quasi-norm. In this section we will show that Sobolev and Besov spaces of dominating mixed smoothness are actually tensor products of corresponding spaces on $\mathbb{R}$. We first recall some notions concerning the tensor products of Banach spaces. We follow [62, Chapter 1], but see also [21, Chapters 1,2]. Let X and Y be Banach spaces. Consider the set of all formal expressions

$$\sum_{i=1}^n f_i \otimes g_i\,, \qquad n\in\mathbb{N}\,, \qquad f_i \in X \quad \text{and} \quad g_i \in Y.$$

We introduce an equivalence relation by means of

$$\sum_{i=1}^n f_i \otimes g_i \sim \sum_{j=1}^m u_j \otimes v_j$$

if both expressions generate the same operator $A:\ X' \to Y$, i.e.,

$$\sum_{i=1}^n \varphi(f_i)\, g_i = \sum_{j=1}^m \varphi(u_j)\, v_j \qquad \text{for all} \quad \varphi \in X'\,.$$

Here recall that X' denotes the dual space of X. The algebraic tensor product $X\otimes Y$ of X and Y is defined to be the set of all such equivalence classes. We wish to equip the set $X\otimes Y$ with some norm. Let X_1, X_2, Y_1, Y_2 be Banach spaces and $T_i \in \mathcal{L}(X_i, Y_i)$, $i = 1, 2$. The tensor product of T_1 and T_2 is defined as

$$(T_1\otimes T_2)h := \sum_{i=1}^n (T_1 f_i)\otimes(T_2 g_i)\,, \qquad h = \sum_{i=1}^n f_i\otimes g_i \in X_1\otimes X_2\,.$$

A norm $\alpha(\cdot, X, Y)$ on $X\otimes Y$ is called a uniform tensor norm if the inequality

$$\alpha\big((T_1\otimes T_2)h, Y_1, Y_2\big) \le \|T_1|\mathcal{L}(X_1,Y_1)\|\cdot\|T_2|\mathcal{L}(X_2,Y_2)\|\ \alpha\big(h, X_1, X_2\big)$$

holds for all

$$h = \sum_{j=1}^n f_j\otimes g_j \in X_1\otimes X_2$$

and all $T_1 \in \mathcal{L}(X_1, Y_1)$, $T_2 \in \mathcal{L}(X_2, Y_2)$. The completion of $X \otimes Y$ with respect to the tensor norm α will be denoted by $X \otimes_\alpha Y$. If α is uniform then $T_1 \otimes T_2$ has a unique extension to $X_1 \otimes_\alpha X_2$ which we again denote by $T_1 \otimes T_2$.

Let us recall the p-nuclear tensor norm and the projective tensor norm for Banach spaces.

Definition 1.36. (i) Let X, Y be Banach spaces. Let $1 \le p \le \infty$ and $1/p + 1/p' = 1$. Assume that

$$h = \sum_{j=1}^{n} f_j \otimes g_j \in X \otimes Y, \qquad f_j \in X,\ g_j \in Y.$$

Then the p-nuclear tensor norm $\alpha_p(\cdot, X, Y)$ is given by

$$\alpha_p(h, X, Y) := \inf\left\{ \Big(\sum_{i=1}^{n} \|f_i|X\|^p \Big)^{1/p} \sup\left\{ \Big(\sum_{i=1}^{n} |\psi(g_i)|^{p'} \Big)^{1/p'} : \psi \in Y', \|\psi|Y'\| \le 1 \right\} \right\},$$

where the infimum is taken over all representations of h.
(ii) The projective tensor norm $\gamma_1(\cdot, X, Y)$ is defined as

$$\gamma_1(h, X, Y) = \inf\left\{ \sum_{j=1}^{n} \|f_j|X\|\, \|g_j|Y\| :\ f_j \in X,\, g_j \in Y,\, h = \sum_{j=1}^{n} f_j \otimes g_j \right\}.$$

Remark 1.37. The p-nuclear tensor norm and projective tensor norm are uniform norms. For further properties of these norms we refer to [62, Chapter 1]. There is another well-known construction of tensor norm for Banach spaces, namely the injective tensor norm $\lambda(\cdot, X, Y)$. We refer to [62, 21] for its definition and basic properties.

Tensor products of quasi-Banach spaces have been studied by Turpin [134], Nitsche [79], and Hansen [46]. The approach for Banach spaces applies to quasi-normed spaces as well provided that X' separates the points in X, i.e., for every $x \in X$, $x \ne 0$, there exists a functional $x' \in X'$ such that $\langle x', x\rangle \ne 0$. The concept of the projective tensor-norm γ_1 can be extended to special quasi-Banach spaces which are continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$. Tensor products of two tempered distributions are given as follows. Let $f \in \mathcal{S}'(\mathbb{R}^{d_1})$ and $g \in \mathcal{S}'(\mathbb{R}^{d_2})$. Then there exists a unique distribution $h \in \mathcal{S}'(\mathbb{R}^{d_1+d_2})$ such that for all $\varphi \in \mathcal{S}(\mathbb{R}^{d_1})$, $\psi \in \mathcal{S}(\mathbb{R}^{d_2})$ we have

$$h(\varphi \otimes \psi) = f(\varphi) \cdot g(\psi).$$

The distribution h is called the tensor product of f and g and is denoted by $f \otimes^D g$.

Definition 1.38. Let $0 < p \le 1$. Let further X and Y be quasi-Banach spaces such that $X \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_1})$ and $Y \hookrightarrow \mathcal{S}'(\mathbb{R}^{d_2})$. Then the projective tensor p-norm γ_p is defined as

$$\gamma_p(h, X, Y) := \inf\left\{ \Big(\sum_{j=1}^{n} \|f_j|X\|^p\, \|g_j|Y\|^p \Big)^{1/p} :\ f_j \in X,\, g_j \in Y,\, h = \sum_{j=1}^{n} f_j \otimes^D g_j \right\}.$$

Tensor products of Besov and Sobolev spaces have been investigated in [46], [106], and [107]. In the following we will recall the results from [106]. We use the notation $\sigma_p = \alpha_p$ if $1 < p < \infty$ and $\sigma_p = \gamma_p$ if $0 < p \le 1$.

Theorem 1.39. *Let $d > 1$, $t \in \mathbb{R}$ and $0 < p < \infty$. Then the following formula*

$$S^t_{p,p}B(\mathbb{R}^d) = B^t_{p,p}(\mathbb{R}) \otimes_{\sigma_p} \ldots \otimes_{\sigma_p} B^t_{p,p}(\mathbb{R}) \quad (d \text{ times})$$

holds true in the sense of equivalent quasi-norms.

Remark 1.40. Tensor products of more than two spaces should be understood as iterated tensor products, i.e., $X \otimes_{\sigma_p} Y \otimes_{\sigma_p} Z = X \otimes_{\sigma_p} (Y \otimes_{\sigma_p} Z)$. By considering the closure of the set of Schwartz functions in the corresponding spaces, Theorem 1.39 can be extended to the case $p = \infty$ with the injective tensor norm. For more details we refer to [106].

Concerning the Sobolev spaces we have the following theorem.

Theorem 1.41. *Let $d > 1$, $t \in \mathbb{R}$ and $1 < p < \infty$. Then the following formula*

$$S^t_pH(\mathbb{R}^d) = H^t_p(\mathbb{R}) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} H^t_p(\mathbb{R}) \quad (d \text{ times})$$

holds true in the sense of equivalent norms.

1.2.3 Dual spaces

The aim of this section is to study the topological dual spaces of Besov and Triebel-Lizorkin spaces of dominating mixed smoothness. For $1 < p < \infty$ the conjugate exponent p' of p is determined by $\frac{1}{p} + \frac{1}{p'} = 1$. If $0 < p \le 1$ we put $p' = \infty$ and $p' = 1$ if $p = \infty$. To study the dual spaces of $S^t_{p,q}A(\mathbb{R}^d)$ it will be convenient for us to switch to the closure of $\mathcal{S}(\mathbb{R}^d)$ in these spaces which will be denoted by $\mathring{S}^t_{p,q}A(\mathbb{R}^d)$. Recall that $\mathring{S}^t_{p,q}A(\mathbb{R}^d) = S^t_{p,q}A(\mathbb{R}^d)$ if and only if $\max(p,q) < \infty$. Because of the density of $\mathcal{S}(\mathbb{R}^d)$ in these spaces, any element of their dual spaces can be interpreted as an element of $\mathcal{S}'(\mathbb{R}^d)$. Hence, a tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to the dual space $[\mathring{S}^t_{p,q}A(\mathbb{R}^d)]'$ if and only if there exists a positive constant C such that

$$|f(\varphi)| \le C\, \|\varphi|\mathring{S}^t_{p,q}A(\mathbb{R}^d)\|$$

holds for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Similarly for the spaces $\mathring{A}^t_{p,q}(\mathbb{R}^d)$. We wish to emphasize that all the statements in this section must be understood in this sense. For later use, let us first recall the results for the isotropic spaces.

Proposition 1.42. *Let $t \in \mathbb{R}$.*
(i) *If $1 \le p < \infty$ and $0 < q \le \infty$, then it holds*

$$[\mathring{B}^t_{p,q}(\mathbb{R}^d)]' = B^{-t}_{p',q'}(\mathbb{R}^d).$$

(ii) *If $0 < p < 1$ and $0 < q \le \infty$, then it holds*

$$[\mathring{B}^t_{p,q}(\mathbb{R}^d)]' = B^{-t+d(\frac{1}{p}-1)}_{\infty,q'}(\mathbb{R}^d).$$

(iii) *If $1 < p < \infty$ and $1 \le q \le \infty$, then it holds*

$$[\mathring{F}^t_{p,q}(\mathbb{R}^d)]' = F^{-t}_{p',q'}(\mathbb{R}^d).$$

(iv) *If $0 < p < 1$ and $0 < q \le \infty$ then it holds*

$$[\mathring{F}^t_{p,q}(\mathbb{R}^d)]' = B^{-t+d(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^d).$$

For a proof of Proposition 1.42 we refer the reader to [129, Section 2.5.1], [130, Section 2.11], and [69]. To prepare for the situation of dominating mixed smoothness we need the following lemmas.

Lemma 1.43. *Let $t \in \mathbb{R}$, $1 \le p \le \infty$ and $0 < q \le \infty$. Let further $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ be the system defined in Remark 1.24. Then the space $S^t_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that there exists a sequence $\{f_k\}_{k\in\mathbb{N}_0^d} \subset L_p(\mathbb{R}^d)$ satisfying*

$$f = \sum_{k\in\mathbb{N}_0^d} \mathcal{F}^{-1}\varphi_k\mathcal{F}f_k \quad \text{in } \mathcal{S}'(\mathbb{R}^d) \qquad \text{and} \quad \|2^{t|k|_1} f_k|\ell_q(L_p)\| < \infty\,. \tag{1.11}$$

The norm

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\|^* := \inf \|2^{t|k|_1} f_k|\ell_q(L_p)\|$$

is equivalent to the norm in (1.7)*. Here the infimum is taken over all admissible representations in* (1.11)*.*

Lemma 1.44. *Let $t \in \mathbb{R}$, $1 < p < \infty$ and $1 \le q \le \infty$. Let further $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ be the system defined in Remark 1.24. Then the space $S^t_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that there exists a sequence $\{f_k\}_{k\in\mathbb{N}_0^d} \subset L_p(\mathbb{R}^d)$ satisfying*

$$f = \sum_{k\in\mathbb{N}_0^d} \mathcal{F}^{-1}\varphi_k\mathcal{F}f_k \quad \text{in } \mathcal{S}'(\mathbb{R}^d) \qquad \text{and} \quad \|2^{t|k|_1} f_k|L_p(\ell_q)\| < \infty\,. \tag{1.12}$$

The norm

$$\|f|S^t_{p,q}F(\mathbb{R}^d)\|^* := \inf \|2^{t|k|_1} f_k|L_p(\ell_q)\|$$

is equivalent to the norm in (1.8)*. Here the infimum is taken over all admissible representations in* (1.12)*.*

We shall give a proof of Lemma 1.44. By using Fourier multiplier assertion for $L_p(\mathbb{R}^d)$, see, e.g., [100, Proposition 2.1.6.4], the proof of Lemma 1.43 can be treated similarly.

Proof of Lemma 1.44. *Step 1.* Let $\{\varphi_j\}_{j=0}^\infty$ be the system given in Remark 1.24. We put

$$\tilde\varphi_j := \varphi_{j-1} + \varphi_j + \varphi_{j+1}\,, \qquad j \in \mathbb{N}_0$$

with $\varphi_{-1} \equiv 0$. If $k \in \mathbb{N}_0^d$ we define $\tilde\varphi_k := \tilde\varphi_{k_1} \otimes \ldots \otimes \tilde\varphi_{k_d}$. For $f \in S^t_{p,q}F(\mathbb{R}^d)$ we choose $f_k = \mathcal{F}^{-1}\tilde\varphi_k\mathcal{F}f$. Then we have

$$\begin{aligned}\|f|S^t_{p,q}F(\mathbb{R}^d)\|^* &\le \|2^{t|k|_1} f_k|L_p(\ell_q)\|\\ &= \big\|2^{t|k|_1}\mathcal{F}^{-1}\tilde\varphi_k\mathcal{F}f\big|L_p(\ell_q)\big\| \le c\,\big\|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}f\big|L_p(\ell_q)\big\|\,.\end{aligned}$$

Step 2. Assume that f can be represented in (1.12). We put $\varphi_k \equiv 0$ if $\min_{i=1,\ldots,d} k_i < 0$. Then we obtain

$$\mathcal{F}^{-1}\varphi_k\mathcal{F}f = \mathcal{F}^{-1}\Big(\varphi_k \sum_{\ell\in\{-1,0,1\}^d} \varphi_{k+\ell}\mathcal{F}f_{k+\ell}\Big).$$

Applying Lemma 1.11 we get

$$\begin{aligned}\|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}f|L_p(\ell_q)\| &= \Big\|2^{t|k|_1}\mathcal{F}^{-1}\Big(\varphi_k\sum_{\ell\in\{-1,0,1\}^d}\varphi_{k+\ell}\mathcal{F}f_{k+\ell}\Big)\Big|L_p(\ell_q)\Big\| \\ &\le c_1\|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}f_k|L_p(\ell_q)\|.\end{aligned}$$

To continue we split $\sum_k$ into several parts. Observe that

$$\sum_{k\in\mathbb{N}_0^d}\big|2^{t|k|_1}\mathcal{F}^{-1}[\varphi_k\mathcal{F}f_k]\big|^q = \sum_{e\subset\{1,\ldots,d\}}\sum_{\substack{k_i\ge1, i\in e\\ k_j=0, j\notin e}}\big|2^{t|k|_1}\mathcal{F}^{-1}(\varphi_k\mathcal{F}f_k)\big|^q. \tag{1.13}$$

Proposition 1.4 can be applied to each subsum to yield

$$\|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}f|L_p(\ell_q)\| \le c_2\|2^{t|k|_1}f_k|L_p(\ell_q)\|$$

with a constant c_2 independent of f. That finishes the proof. ∎

Proposition 1.45. *Let $t\in\mathbb{R}$.*
(i) *If $1\le p<\infty$ and $0<q\le\infty$, then it holds*

$$[\mathring{S}^t_{p,q}B(\mathbb{R}^d)]' = S^{-t}_{p',q'}B(\mathbb{R}^d).$$

(ii) *If $0<p<1$ and $0<q\le\infty$, then it holds*

$$[\mathring{S}^t_{p,q}B(\mathbb{R}^d)]' = S^{-t+\frac{1}{p}-1}_{\infty,q'}B(\mathbb{R}^d).$$

Proof. For the proof, at least if $0<p,q<\infty$, we refer to Hansen [45, Section 2.3.8]. Here we only give a proof in case $q=\infty$. We follows essentially the arguments given in [129, Section 2.5.1] for the isotropic spaces.
Step 1. Proof of (i).
Substep 1.1. Let $f\in S^{-t}_{p',1}B(\mathbb{R}^d)$. Then we can find $\{f_k\}_{k\in\mathbb{N}_0^d}\subset L_{p'}(\mathbb{R}^d)$ such that

$$f=\sum_{k\in\mathbb{N}_0^d}\mathcal{F}^{-1}\varphi_k\mathcal{F}f_k \ \text{ in } \mathcal{S}'(\mathbb{R}^d) \quad\text{and}\quad \|2^{-t|k|_1}f_k|\ell_1(L_{p'})\|\le 2\|f|S^{-t}_{p',1}B(\mathbb{R}^d)\|^*.$$

If $\varrho\in\mathcal{S}(\mathbb{R}^d)$, from the symmetry property of $\{\varphi_k\}_k$, see Remark 1.24, we have

$$\begin{aligned}|f(\varrho)| &= \Big|\sum_{k\in\mathbb{N}_0^d}(\mathcal{F}^{-1}\varphi_k\mathcal{F}f_k)(\varrho)\Big| = \Big|\sum_{k\in\mathbb{N}_0^d}f_k(\mathcal{F}\varphi_k\mathcal{F}^{-1}\varrho)\Big| \\ &\le c_1\|2^{-t|k|_1}f_k|\ell_1(L_{p'})\|\cdot\|2^{t|k|_1}\mathcal{F}\varphi_k\mathcal{F}^{-1}\varrho|\ell_\infty(L_p)\| \\ &= c_1\|2^{-t|k|_1}f_k|\ell_1(L_{p'})\|\cdot\|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}\varrho|\ell_\infty(L_p)\| \\ &\le c_2\|f|S^{-t}_{p',1}B(\mathbb{R}^d)\|\cdot\|\varrho|S^t_{p,\infty}B(\mathbb{R}^d)\|\end{aligned}$$

which implies $f\in[\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'$.
Substep 1.2. Next we prove the reverse direction. Let $c_0(L_p)$ denote the space of all sequences $\{\psi_k\}_{k\in\mathbb{N}_0^d}$ of measurable functions such that

$$\lim_{|k|_1\to\infty}\|\,\psi_k\,|L_p(\mathbb{R}^d)\|=0$$

equipped with the norm

$$\| \psi_k \,|c_0(L_p)\| := \sup_{k\in\mathbb{N}_0^d} \| \psi_k \,|L_p(\mathbb{R}^d)\| \,.$$

Observe that the mapping

$$J: \; g \mapsto \{2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}g\}_{k\in\mathbb{N}_0^d}$$

is isometric and bijective if J is considered as a mapping from $\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)$ onto a closed subspace Y of $c_0(L_p)$. Here we use the fact that

$$\lim_{|k|_1\to\infty} \| 2^{t|k|_1}\, \mathcal{F}^{-1}\varphi_k\mathcal{F}g \,|L_p(\mathbb{R}^d)\| = 0$$

holds for all $g \in \mathring{S}^t_{p,\infty}B(\mathbb{R}^d)$.

Let $f \in [\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'$. Hence, by defining

$$\tilde{f}\big(\{\psi_k\}_k\big) := f\Big(\sum_{k\in\mathbb{N}_0^d} 2^{-t|k|_1}\,\psi_k\Big)\,, \qquad \{\psi_k\}_k \in Y\,,$$

$\tilde{f}$ becomes a linear and continuous functional on Y satisfying

$$\| \tilde{f} \,|Y \to \mathbb{C}\| = \| f \,|\, [\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\|.$$

Now, by the Hahn-Banach theorem, there exists a linear and continuous extension of $\tilde{f}$ to a continuous linear functional on the space $c_0(L_p)$ with preservation of norm. It is known that $[c_0(L_p)]' = \ell_1(L_{p'})$ and any $g \in [c_0(L_p)]'$ can be represented in the form

$$g(\{\psi_k\}_k) = \sum_{k\in\mathbb{N}_0^d}\int_{\mathbb{R}^d} g_k(x)\,\psi_k(x)\;\mathrm{d}x\,, \qquad \{\psi_k\}_k \in c_0(L_p)\,, \tag{1.14}$$

where the functions g_k satisfy

$$\| g \,|\ell_1(L_{p'})\| = \sum_{k\in\mathbb{N}_0^d} \|g_k \,|L_{p'}(\mathbb{R}^d)\| < \infty\,,$$

see [128, Lemma 1.11.1]. Applying this with $g = \tilde{f}$ we find

$$\| f_k \,|\ell_1(L_{p'})\| = \| \tilde{f} \,|[c_0(L_p)]'\| = \| f \,|\, [\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\| \tag{1.15}$$

for an appropriate sequence $\{f_k\}_k$. In view of (1.14), the definition of $\tilde{f}$ and the symmetry condition with respect to $\{\varphi_k\}_k$ we obtain

$$\begin{aligned} f(\psi) &= f\Big(\sum_{k\in\mathbb{N}_0^d} \mathcal{F}^{-1}\varphi_k\mathcal{F}\psi\Big) = \tilde{f}\Big(\{2^{t|k|_1}\,\mathcal{F}^{-1}\varphi_k\mathcal{F}\psi\}_k\Big) \\ &= \sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1} \int_{\mathbb{R}^d} f_k(x)\,(\mathcal{F}^{-1}\varphi_k\mathcal{F}\psi)(x)\;\mathrm{d}x = \sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1}\,\big(\mathcal{F}^{-1}\varphi_k\mathcal{F}f_k\big)(\psi) \end{aligned}$$

for any $\psi \in \mathcal{S}(\mathbb{R}^d)$. This leads to the identity

$$f = \sum_{k \in \mathbb{N}_0^d} 2^{t|k|_1} \, \mathcal{F}^{-1}\big(\varphi_k \mathcal{F} f_k\big)$$

valid in $\mathcal{S}'(\mathbb{R}^d)$. In view of Lemma 1.43 and (1.15) we arrive at

$$\|f|S^{-t}_{p',1}B(\mathbb{R}^d)\| \le c_1 \|f_k|\ell_1(L_{p'})\| \;=\; \|\, f \,|\, [\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\|$$

which proves $f \in S^{-t}_{p',1}B(\mathbb{R}^d)$.
Step 2. We prove (ii).
Substep 2.1. It follows from Lemma 1.34 that $S^t_{p,\infty}B(\mathbb{R}^d) \hookrightarrow S^{t-\frac{1}{p}+1}_{1,\infty}B(\mathbb{R}^d)$ which implies $\mathring{S}^t_{p,\infty}B(\mathbb{R}^d) \hookrightarrow \mathring{S}^{t-\frac{1}{p}+1}_{1,\infty}B(\mathbb{R}^d)$. Duality and Step 1 yield

$$S^{-t+\frac{1}{p}-1}_{\infty,1}B(\mathbb{R}^d) \hookrightarrow [\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]' \,.$$

Substep 2.2. Let $f \in [\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'$. We follow the argument in Hansen [45, pages 75, 76]. For any $k \in \mathbb{N}_0^d$ we choose a point $x_k \in \mathbb{R}^d$ such that

$$\frac{1}{2}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|L_\infty(\mathbb{R}^d)\| \le |(\mathcal{F}^{-1}\varphi_k\mathcal{F}f)(x_k)| \le \|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|L_\infty(\mathbb{R}^d)\| \,. \tag{1.16}$$

Then we define the function

$$\psi(x) := \sum_{|\ell|_1 \le n} a_\ell (\mathcal{F}^{-1}\varphi_\ell)(x_\ell - x)\, 2^{|\ell|_1(-t+\frac{1}{p}-1)} \,, \qquad x \in \mathbb{R}^d \,.$$

Obviously $\psi \in \mathcal{S}(\mathbb{R}^d)$. An easy calculation yield

$$\begin{aligned}
&\|\,\psi\,|S^t_{p,\infty}B(\mathbb{R}^d)\|^p \\
&= \sup_{k \in \mathbb{N}_0^d} 2^{t|k|_1 p} \Bigg\| \mathcal{F}^{-1}\Big(\sum_{\substack{\ell \in \{-1,0,1\}^d \\ |k+\ell|_1 \le n}} 2^{|k+\ell|_1(-t+\frac{1}{p}-1)} a_{k+\ell}\varphi_k(y)\varphi_{k+\ell}(-y)e^{-ix_{(k+\ell)}y}\Big)(\cdot) \Bigg| L_p(\mathbb{R}^d) \Bigg\|^p \\
&\le \sup_{k \in \mathbb{N}_0^d} 2^{t|k|_1 p} \sum_{\substack{\ell \in \{-1,0,1\}^d \\ |k+\ell|_1 \le n}} \big\| 2^{|k+\ell|_1(-t+\frac{1}{p}-1)}\, a_{k+\ell}\, \mathcal{F}^{-1}[\varphi_k\varphi_{k+\ell}](\cdot)\, \big| L_p(\mathbb{R}^d) \big\|^p \\
&\le c_1 \sup_{k \in \mathbb{N}_0^d} 2^{t|k|_1 p} \sum_{\substack{\ell \in \{-1,0,1\}^d \\ |k+\ell|_1 \le n}} \big\| 2^{|k+\ell|_1(-t+\frac{1}{p}-1)}\, a_{k+\ell}\, \mathcal{F}^{-1}\varphi_{k+\ell}(\cdot)\, \big| L_p(\mathbb{R}^d) \big\|^p,
\end{aligned}$$

where the last inequality is a consequence of Lemma 1.6 and a homogeneity argument. Observe that

$$\|\mathcal{F}^{-1}\varphi_{k+\ell}(\cdot)\,|L_p(\mathbb{R}^d)\| = \|\,\mathcal{F}\varphi_{k+\ell}\,|L_p(\mathbb{R}^d)\| = 2^{|k+\ell|_1(1-\frac{1}{p})}\,\|\,\mathcal{F}\varphi_{\bar{1}}\,|L_p(\mathbb{R}^d)\|$$

if $k+\ell \ge \bar{1}$. In case $\min_{i=1,\ldots,d}(k_i+\ell_i) = 0$ one has to modify this in an obvious way. Altogether we have found

$$\|\,\psi\,|S^t_{p,\infty}B(\mathbb{R}^d)\| \le c_2 \sup_{k \in \mathbb{N}_0^d} \Bigg(\sum_{\substack{\ell \in \{-1,0,1\}^d \\ |k+\ell|_1 \le n}} |a_{k+\ell}|^p \Bigg)^{1/p} \le c_3 \sup_{|k|_1 \le n} |a_k| \,,$$

with c_3 independent of n. This estimate can be used to derive

$$\begin{aligned}\Big|\sum_{|k|_1\le n} a_k\, 2^{|k|_1(-t+\frac{1}{p}-1)}(\mathcal{F}^{-1}\varphi_k\mathcal{F}f)(x_k)\Big| &= \Big|\sum_{|k|_1\le n} a_k 2^{|k|_1(-t+\frac{1}{p}-1)}(f*\mathcal{F}^{-1}\varphi_k)(x_k)\Big|\\ &= |f(\psi)|\\ &\le \|f\,|[\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\|\cdot\|\psi|S^t_{p,\infty}B(\mathbb{R}^d)\|\\ &\le c_3\,\|f\,|[\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\,\|\cdot\sup_{|k|_1\le n}|a_k|\,.\end{aligned}$$

Employing (1.16) and the fact that the a_k can be chosen as we want, for instance, such that

$$a_k\,(\mathcal{F}^{-1}\varphi_k\mathcal{F}f)(x_k) = |(\mathcal{F}^{-1}\varphi_k\mathcal{F}f)(x_k)|\,,$$

we find $|a_k| = 1$ for all k with $|k|_1 \le n$ and hence

$$\sum_{|k|_1\le n} 2^{|k|_1(-t+\frac{1}{p}-1)}\,\|\,\mathcal{F}^{-1}\varphi_k\mathcal{F}f\,|L_\infty(\mathbb{R}^d)\| \le c_3\,\|f\,|[\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\,\|\,.$$

Here c_3 is independent of f and n. For $n\to\infty$ we obtain

$$\|\,f\,|S^{-t+\frac{1}{p}-1}_{\infty,1}B(\mathbb{R}^d)\| \le c_3\,\|f\,|[\mathring{S}^t_{p,\infty}B(\mathbb{R}^d)]'\,\|\,.$$

This finishes the proof. ■

Let $L_p(c_0)$ denote the space of all sequences $\{\psi_k\}_{k\in\mathbb{N}_0^d}$ of measurable functions such that $\lim_{|k|_1\to\infty}|\psi_k(x)| = 0$ for almost $x\in\mathbb{R}^d$ equipped with the norm

$$\|\psi_k|L_p(c_0)\| := \Big\|\sup_{k\in\mathbb{N}_0^d}|\psi_k(\cdot)|\Big|L_p(\mathbb{R}^d)\Big\|.$$

We shall need the following lemma, see Triebel [130, Proposition 2.11.1] and [34, Theorems 8.18.2, 8.20.3].

Lemma 1.46. (i) *Let $1\le p<\infty$ and $0<q<\infty$. Then $g\in[L_p(\ell_q)]'$ if and only if it can be represented uniquely as*

$$g(f) = \sum_{k\in\mathbb{N}_0^d}\int_{\mathbb{R}^d} g_k f_k(x)\,\mathrm{d}x$$

for every $f=\{f_k\}_{k\in\mathbb{N}_0^d}\in L_p(\ell_q)$, where

$$g=\{g_k\}_{k\in\mathbb{N}_0^d}\in L_{p'}(\ell_{q'})\quad\textit{and}\quad \|g\| = \|g_k|L_{p'}(\ell_{q'})\|\,.$$

(ii) *Let $1<p<\infty$. Then we have*

$$(L_p(c_0))' = L_{p'}(\ell_1).$$

Proposition 1.47. *Let $t \in \mathbb{R}$.*
(i) *If $1 < p < \infty$ and $1 \le q \le \infty$, then it holds*

$$[\mathring{S}^t_{p,q}F(\mathbb{R}^d)]' = S^{-t}_{p',q'}F(\mathbb{R}^d).$$

(ii) *If $0 < p < 1$ and $0 < q \le \infty$, then it holds*

$$[\mathring{S}^t_{p,q}F(\mathbb{R}^d)]' = S^{-t+\frac{1}{p}-1}_{\infty,\infty}B(\mathbb{R}^d).$$

Proof. A part of the result in (i), i.e., $1 < p, q < \infty$, has been proved by Hansen [45, Section 5.5] for sequence spaces associated with Triebel-Lizorkin spaces of dominating mixed smoothness. Employing Lemmas 1.44, 1.46 and the similar argument as Step 1 in the proof of Proposition 1.45 we obtain part (i) for $q = 1$ and $q = \infty$ as well. We prove part (ii). We have from Lemma 1.34

$$\mathring{S}^t_{p,\min(p,q)}B(\mathbb{R}^d) \hookrightarrow \mathring{S}^t_{p,q}F(\mathbb{R}^d) \hookrightarrow \mathring{S}^{t-\frac{1}{p}+1}_{1,1}F(\mathbb{R}^d) = \mathring{S}^{t-\frac{1}{p}+1}_{1,1}B(\mathbb{R}^d)\,.$$

Now duality yields

$$S^{-t+\frac{1}{p}-1}_{\infty,\infty}B(\mathbb{R}^d) \hookrightarrow [\mathring{S}^t_{p,q}F(\mathbb{R}^d)]' \hookrightarrow S^{-t+\frac{1}{p}-1}_{\infty,\infty}B(\mathbb{R}^d)\,.$$

This finishes the proof. ■

1.2.4 Complex interpolation

First we briefly describe the complex interpolation method following the original paper [16] and the monographs [11, 66, 128]. In the meanwhile it is well-known that this complex interpolation method extends to specific quasi-Banach spaces, namely those, which are analytically convex. The analytically convex condition ensures that the Maximum Modulus Principle is valid, see [72] and the references given there for details. Let us recall that a quasi-Banach space X is called analytically convex if there is a constant C such that for every polynomial $P : \mathbb{C} \to X$ we have

$$\|P(0)|X\| \le C \max_{|z|=1} \|P(z)|X\|.$$

Note that any Banach space is analytically convex. Let X_0 and X_1 be two quasi-Banach spaces. If X_0, X_1 are continuously embedded in a Hausdorff topological vector space $\mathcal{V}$ then we say that X_0, X_1 form an interpolation couple (X_0, X_1). Given an interpolation couple (X_0, X_1) we define the space

$$X_0 + X_1 = \big\{x \in \mathcal{V} : \|x|X_0 + X_1\| < \infty\big\}$$

where

$$\|x|X_0 + X_1\| = \inf\big\{\|x_0|X_0\| + \|x_1|X_1\| :\ x = x_0 + x_1,\ x_i \in X_i,\ i = 0, 1\big\}.$$

Let S be the strip $S = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1\}$. By $\mathfrak{F}$ we denote the class of all functions $f :\ \bar{S} \to X_0 + X_1$ which is bounded, continuous on the closure $\bar{S}$ of S and analytic on S.

Moreover, the functions $t \to f(j+it)$, $j = 0,1$, are bounded continuous functions from $\mathbb{R}$ into X_j. We equip the class $\mathfrak{F}$ with the quasi-norm

$$\| f \,|\, \mathfrak{F} \| = \max \Big\{ \sup_{t \in \mathbb{R}} \|f(it)|X_0\|, \ \sup_{t \in \mathbb{R}} \|f(1+it)|X_1\| \Big\}.$$

Finally, the interpolation space is defined as

$$[X_0, X_1]_\Theta := \big\{ x \in X_0 + X_1 : x = f(\Theta) \text{ for some } f \in \mathfrak{F} \big\}, \quad 0 < \Theta < 1$$

and endowed with the quasi-norm

$$\| x \,|\, [X_0, X_1]_\Theta \| := \inf \big\{ \| f \,|\, \mathfrak{F} \| : f \in \mathfrak{F}, \ f(\Theta) = x \big\}.$$

The following proposition, well-known in case of Banach spaces, see [11, Theorem 4.1.2], [66, Theorem 2.1.6] or [128, Theorem 1.10.3.1], can also be extended to the quasi-Banach case, see [52].

Proposition 1.48. *Let $0 < \Theta < 1$. Let (X_0, X_1) and (Y_0, Y_1) be two compatible couples of quasi-Banach spaces. In addition, let $X_0 + X_1$, $Y_0 + Y_1$ be analytically convex. If T is in $\mathcal{L}(X_0, Y_0)$ and in $\mathcal{L}(X_1, Y_1)$, then the restriction of T to $[X_0, X_1]_\Theta$ is in $\mathcal{L}([X_0, X_1]_\Theta, [Y_0, Y_1]_\Theta)$ for every Θ. Moreover,*

$$\|T : [X_0, X_1]_\Theta \to [Y_0, Y_1]_\Theta\| \ \le \ \|T : X_0 \to Y_0\|^{1-\Theta} \, \|T : X_1 \to Y_1\|^{\Theta}.$$

Vybiral [140, Theorem 4.6] has proved the following result for sequence spaces associated to Besov and Triebel-Lizorkin spaces of dominating mixed smoothness. However, these results can be shifted to the level of function spaces by suitable wavelet isomorphisms, see [140, Theorem 2.12].

Proposition 1.49. *Let $t_i \in \mathbb{R}$, $0 < p_i, q_i \le \infty$, $i = 0,1$ and $\Theta \in (0,1)$. Let further t, p and q be given by*

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1}, \qquad \frac{1}{q} = \frac{1-\Theta}{q_0} + \frac{\Theta}{q_1}, \qquad t = (1-\Theta)t_0 + \Theta t_1. \tag{1.17}$$

(i) *If* $\min\big(\max(p_0,q_0), \max(p_1,q_1)\big) < \infty$, *then*

$$S^t_{p,q}B(\mathbb{R}^d) = [S^{t_0}_{p_0,q_0}B(\mathbb{R}^d), \ S^{t_1}_{p_1,q_1}B(\mathbb{R}^d)]_\Theta.$$

(ii) *If* $0 < p_0, p_1 < \infty$ *and* $\min(q_0, q_1) < \infty$, *then*

$$S^t_{p,q}F(\mathbb{R}^d) = [S^{t_0}_{p_0,q_0}F(\mathbb{R}^d), \ S^{t_1}_{p_1,q_1}F(\mathbb{R}^d)]_\Theta.$$

Remark 1.50. Observe that the conditions in (i) and (ii) imply that at least one of the spaces in the interpolation couple $(S^{t_0}_{p_0,q_0}A(\mathbb{R}^d), S^{t_1}_{p_1,q_1}A(\mathbb{R}^d))$ is separable. These assertions do not extend to the case where both spaces are not separable.

Concerning the isotropic setting we have the analogous results.

Proposition 1.51. *Let $t_i \in \mathbb{R}$, $0 < p_i, q_i \leq \infty$, $i = 0, 1$ and $\Theta \in (0,1)$. Let further t, p and q be given by* (1.17).
(i) *If* $\min\big(\max(p_0, q_0), \max(p_1, q_1)\big) < \infty$ *then*

$$B^t_{p,q}(\mathbb{R}^d) = [B^{t_0}_{p_0,q_0}(\mathbb{R}^d),\ B^{t_1}_{p_1,q_1}(\mathbb{R}^d)]_\Theta.$$

(ii) *If* $0 < p_0, p_1 < \infty$ *and* $\min(q_0, q_1) < \infty$*, then*

$$F^t_{p,q}(\mathbb{R}^d) = [F^{t_0}_{p_0,q_0}(\mathbb{R}^d),\ F^{t_1}_{p_1,q_1}(\mathbb{R}^d)]_\Theta.$$

Remark 1.52. Complex interpolation of isotropic Besov and Lizorkin-Triebel spaces has been studied at various places, we refer to [11, Theorem 6.4.5], [128, Section 2.4.1], [130, Theorem 2.4.7] and [36, 72, 52] as well as to the references given there. In particular we refer to [144] if both spaces in the interpolation couple $(A^{t_0}_{p_0,q_0}(\mathbb{R}^d), A^{t_1}_{p_1,q_1}(\mathbb{R}^d))$ are not separable.

1.2.5 Characterization by mixed iterated differences

For us it will be convenient to study characterizations of Besov and Triebel-Lizorkin spaces of dominating mixed smoothness by differences which is actually the classical way of defining them, see for instance Nikol'skij [78], Amanov [1], Schmeißer, Triebel [104], Temlyakov [120], and the references given there. Let us first recall the basic concepts. Let $i \in [d]$, $m \in \mathbb{N}$, $h \in \mathbb{R}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we put

$$\Delta^m_{h,j} f(x) := \sum_{\ell=0}^{m} (-1)^{m-\ell} \binom{m}{\ell} f(x_1, \ldots, x_{j-1}, x_j + \ell h, x_{j+1}, \ldots, x_d)\,.$$

This is the m-th order differences of f in direction j. For $e \subset [d]$, $h \in \mathbb{R}^d$ and $m \in \mathbb{N}^d_0$ the mixed (m,e)-th differences operator $\Delta^{m,e}_h$ is defined as

$$\Delta^{m,e}_h := \prod_{i \in e} \Delta^{m_i}_{h_i,i} \qquad \text{and} \qquad \Delta^{m,\emptyset}_h := \mathrm{Id}\,,$$

where $\mathrm{Id}\, f = f$. Let us further define the mixed (m,e)th modulus of continuity by

$$\omega^e_m(f,s)_p := \sup_{|h_i| < s_i, i \in e} \|\Delta^{m,e}_h f(\cdot) | L_p(\mathbb{R}^d)\|\,, \qquad s \in [0,1]^d,$$

for $f \in L_p(\mathbb{R}^d)$ (in particular, $\omega^\emptyset_m(f,s)_p = \|f|L_p(\mathbb{R}^d)\|$). We continue by introducing the so-called rectangular means of differences which is the counterpart of the ball means of differences for isotropic spaces, see [130, Theorem 2.5.11]. The rectangular means of differences is defined as

$$\mathcal{R}^e_m(f,s,x) := \int\limits_{[-1,1]^d} |\Delta^{m,e}_{h \diamond s} f(x)|\,\mathrm{d}h\,, \quad x \in \mathbb{R}^d, \quad s \in (0,1]^d\,.$$

We have the following theorems.

Theorem 1.53. *Let $1 \le p \le \infty$, $0 < q \le \infty$ and $t > 0$. Let further $m \in \mathbb{N}_0$ such that $m > t$. Then the space $S^t_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in L_p(\mathbb{R}^d)$ such that*

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\|^{(m)} := \sum_{e\subset[d]} \Big(\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q} \omega^e_{\bar{m}}(f, 2^{-k})^q_p \Big)^{1/q}$$

is finite. The quasi-norm $\|f|S^t_{p,q}B(\mathbb{R}^d)\|^{(m)}$ is equivalent to the quasi-norm in (1.7).

Theorem 1.54. *Let $0 < p < \infty$, $0 < q \le \infty$ and $t > (\frac{1}{\min(p,q)} - 1)_+$. Let further $m \in \mathbb{N}$ such that $m > t$. Then the space $S^t_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in L_p(\mathbb{R}^d)$ such that*

$$\|f|\, S^t_{p,q}F(\mathbb{R}^d)\|^{(m)} := \sum_{e\subset[d]} \Big\| \Big(\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q} \mathcal{R}^e_{\bar{m}}(f, 2^{-k}, \cdot)^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\|$$

is finite. The quasi-norm $\|f|\, S^t_{p,q}F(\mathbb{R}^d)\|^{(m)}$ is equivalent to the quasi-norm in (1.8).

Remark 1.55. The statements in Theorems 1.53 and 1.54 are "discretized versions" of the characterizations given in [104, Section 2.3] and [137]. An extension to $0 < p < 1$ in Theorem 1.53 is possible, but then the smoothness has to be large enough, i.e., $t > \frac{1}{p}$. If we use rectangular means of differences to characterize the spaces $S^t_{p,q}B(\mathbb{R}^d)$, then the condition $p > 1$ can be relaxed to $p > 0$ and $t > (\frac{1}{p} - 1)_+$. The above assertions still hold true if we replace $\bar{m}$ by $m \in \mathbb{N}_0^d$ with $m_i > t$ for all $i = 1, \ldots, d$. For further characterizations by differences of the spaces $S^t_{p,q}A(\mathbb{R}^d)$ we refer again to [104, Section 2.3] and [137].

Remark 1.56. Of peculiar interest is the version of Hölder-Zygmund spaces $\mathcal{Z}^t_{\mathrm{mix}}(\mathbb{R}^d)$. Let $t > 0$. Let further $m \in \mathbb{N}$ such that $m > t$. Then $f \in \mathcal{Z}^t_{\mathrm{mix}}(\mathbb{R}^d)$ if

$$\|\, f\,|\mathcal{Z}^t_{\mathrm{mix}}(\mathbb{R}^d)\| := \|f|C(\mathbb{R}^d)\| + \sum_{e\subset[d], e\neq\emptyset} \sup_{h\in[-1,1]^d} \sup_{x\in\mathbb{R}^d} \prod_{i\in e} |h_i|^{-t} \big|\Delta^{\bar{m},e}_h f(x)\big| < \infty\,.$$

From the characterization of the spaces $S^t_{p,q}B(\mathbb{R}^d)$ we obtain immediately $S^t_{\infty,\infty}B(\mathbb{R}^d) = \mathcal{Z}^t_{\mathrm{mix}}(\mathbb{R}^d)$.

In the following we shall establish the relation $\|f|\, S^t_{p,q}F(\mathbb{R}^d)\| \asymp \|f|S^t_{p,q}F(\mathbb{R}^d)\|^{(m)}$ which is done by using intrinsic characterization of $S^t_{p,q}F(\mathbb{R}^d)$ via local means, see [140], [138]. The proof of Theorem 1.53 follows essentially in the same way but is less technical. Given a function $\Psi : \mathbb{R}^d \to \mathbb{C}$, we denote by $L_\Psi \in \mathbb{N}$ the number of vanishing moments of Ψ, i.e.,

$$\int\limits_{\mathbb{R}} \xi^\alpha \Psi(\xi)\, \mathrm{d}\xi = 0\,, \quad \alpha = 0, \ldots, L_\Psi\,.$$

Let $\Psi_0 \in \mathcal{S}(\mathbb{R})$ be a function satisfying the following conditions

$$\int\limits_{\mathbb{R}} \Psi_0(\xi)\, \mathrm{d}\xi \neq 0 \quad \text{and} \quad L_\Psi \ge R \;\text{ for }\; \Psi(t) = \Psi_0(\xi) - \frac{1}{2}\Psi_0(\xi/2)\,, \tag{1.18}$$

for some $R \in \mathbb{N}$. For $j \in \mathbb{N}$ we put $\Psi_j(\xi) = 2^j \Psi(2^j \xi)$, $\xi \in \mathbb{R}$. If $k \in \mathbb{N}_0^d$ we denote

$$\Psi_k(x) = \Psi_{k_1}(x_1) \cdot \ldots \cdot \Psi_{k_d}(x_d)\,, \quad x \in \mathbb{R}^d\,.$$

We have the following proposition, see [140, Theorem 1.23] and [138].

Proposition 1.57. *Let $0 < p, q \le \infty$ and $t \in \mathbb{R}$. Let further Ψ_0 be given by* (1.18) *with $R+1 > t$.*
(i) *Then the space $S^t_{p,q}B(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\| = \Big(\sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1 q}\|\Psi_k * f|L_p(\mathbb{R}^d)\|^q\Big)^{1/q} < \infty\,.$$

(ii) *Let $0 < p < \infty$. Then the space $S^t_{p,q}F(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|f|\, S^t_{p,q}F(\mathbb{R}^d)\| = \Big\|\Big(\sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1 q}|\Psi_k * f|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| < \infty\,.$$

Based on this result we are in position to prove Theorem 1.54.

Proof of Theorem 1.54. *Step 1.* We prove the inequality

$$\|f|S^t_{p,q}F(\mathbb{R}^d)\|^{(m)} \lesssim \|f|\, S^t_{p,q}F(\mathbb{R}^d)\|\,. \tag{1.19}$$

This inequality is a consequence of [137, Theorem 3.4.1]. Indeed, from the equivalent norm

$$\|f|\, S^t_{p,q}F(\mathbb{R}^d)\| \asymp \sum_{e\subset[d]}\Big\|\Big(\int_{(0,\infty)^{|e|}} \Big\{\Big(\prod_{i\in e} s_i^{-t}\Big)\mathcal{R}^e_{\bar{m}}(f,s,\cdot)\Big\}^q \prod_{i\in e}\frac{\mathrm{d}s_i}{s_i}\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|,$$

see [137, Theorem 3.4.1]. Here, if $e=\emptyset$, the term under the sum is $\|f|L_p(\mathbb{R}^d)\|$. From this we obtain

$$\sum_{e\subset[d]}\Big\|\Big(\int_{(0,1)^{|e|}} \Big\{\Big(\prod_{i\in e} s_i^{-t}\Big)\mathcal{R}^e_{\bar{m}}(f,s,\cdot)\Big\}^q \prod_{i\in e}\frac{\mathrm{d}s_i}{s_i}\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| \lesssim \|f|\, S^t_{p,q}F(\mathbb{R}^d)\|\,.$$

Discretizing the left-hand side of the above inequality the claim follows.
Step 2. We prove the reverse inequality of (1.19). This time we rely on the intrinsic characterization of $S^t_{p,q}F(\mathbb{R}^d)$ via local means. According to Triebel [131, Section 3.3], for $m \in \mathbb{N}$ there exists a function $\psi \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\psi \subset \big(-\frac{1}{m},\frac{1}{m}\big)$ and $\int_{\mathbb{R}}\psi(\xi)\,\mathrm{d}\xi = 1$ such that the function

$$\Psi_0(\xi) = \frac{(-1)^{m+1}}{m!}\sum_{u=1}^m\sum_{v=1}^m (-1)^{u+v}\binom{m}{u}\binom{m}{v} v^m (uv)^{-1}\psi\Big(\frac{\xi}{uv}\Big)\,, \quad \xi\in\mathbb{R}\,,$$

satisfies (1.18) with $R = m$. Putting

$$\eta(\xi) = \frac{(-1)^{m+1}}{m!}\sum_{v=1}^m (-1)^{m-v}\binom{m}{v} v^m v^{-1}\psi\Big(\frac{\xi}{v}\Big)\,, \quad \xi\in\mathbb{R}\,,$$

then we have $\operatorname{supp}\eta \subset (-1,1)$ and moreover

$$\Psi_0(\xi) = \sum_{u=1}^m (-1)^{m-u}\binom{m}{u} u^{-1}\eta\Big(\frac{\xi}{u}\Big)\,.$$

A simple computation gives for a univariate function g

$$\begin{aligned}\Psi_0 * g(\xi) &= \int\limits_{\mathbb{R}} \Psi_0(h) g(\xi - h)\ \mathrm{d}h = \int\limits_{\mathbb{R}} \eta(-h) \sum_{u=1}^{m} (-1)^{m-u} \binom{m}{u} g(\xi + uh)\, \mathrm{d}h \\ &= \int\limits_{-1}^{1} \eta(-h) \big[\Delta_h^m g(\xi) - (-1)^m g(\xi)\big]\, \mathrm{d}h\end{aligned} \tag{1.20}$$

and

$$\begin{aligned}\Psi_j * g(\xi) &= \int\limits_{\mathbb{R}} \Psi(h) g(\xi - 2^{-j}h)\, \mathrm{d}h = \int\limits_{\mathbb{R}} \Big[\eta(-h) - \frac{1}{2}\eta\Big(-\frac{h}{2}\Big)\Big] \Delta_{2^{-j}h}^m g(\xi)\, \mathrm{d}h \\ &= \int\limits_{-1}^{1} \eta(-h) \big[\Delta_{2^{-j}h}^m - \Delta_{2^{-j+1}h}^m\big] g(\xi)\, \mathrm{d}h\, .\end{aligned} \tag{1.21}$$

Next we define the function $\eta_d(\cdot)$ on $\mathbb{R}^d$ as $\eta_d(x) = \eta(x_1)\cdot \ldots \cdot \eta(x_d)$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $e \subset [d]$. For

$$k \in \mathbb{N}^d(e) := \{\ell \in \mathbb{N}_0^d(e),\ \ \ell_i \geq 1,\, i \in e\}$$

from (1.20) and (1.21) we have

$$\Psi_k * f(x) = \int\limits_{[-1,1]^d} \eta_d(-h) \Big[\prod_{i\in e} \big(\Delta_{2^{-k_i}h_i,i}^m - \Delta_{2^{-k_i+1}h_i,i}^m\big) \prod_{i \in e_0} \big(\Delta_{h_i,i}^m - (-1)^m \mathrm{Id}\big)\Big] f(x)\, \mathrm{d}h.$$

Here $e_0 = d\backslash e$. Consequently we obtain

$$|\Psi_k * f(x)|\ \leq \sum_{e_1 : e \subset e_1} \sum_{u \in \{0,1\}^d} \mathcal{R}_{\bar{m}}^{e_1}(f, 2^{-k+u}, x)$$

which leads to

$$\begin{aligned}\sum_{k\in\mathbb{N}^d(e)} 2^{t|k|_1 q} |\Psi_k * f(x)|^q &\leq \sum_{k \in \mathbb{N}^d(e)} 2^{t|k|_1 q} \Big(\sum_{e_1 : e \subseteq e_1} \sum_{u\in\{0,1\}^d} \mathcal{R}_{\bar{m}}^{e_1}(f, 2^{-k+u}, x)\Big)^q \\ &\lesssim \sum_{v\subset[d]} \sum_{k \in \mathbb{N}_0^d(v)} 2^{t|k|_1 q} \mathcal{R}_{\bar{m}}^{v}(f, 2^{-k}, x)^q\, .\end{aligned}$$

If $k = \bar{0}$ we have a similar estimate. From this we conclude that

$$\begin{aligned}\Big\|\Big(\sum_{k\in\mathbb{N}_0^d} 2^{t|k|_1 q} |\Psi_k * f|^q\Big)^{1/q}\Big| L_p(\mathbb{R}^d)\Big\| &\lesssim \Big\|\Big(\sum_{e\subset[d]} \sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q} \mathcal{R}_{\bar{m}}^{e}(f, 2^{-k}, \cdot)^q\Big)^{1/q}\Big| L_p(\mathbb{R}^d)\Big\| \\ &\lesssim \sum_{e\subset[d]} \Big\|\Big(\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q} \mathcal{R}_{\bar{m}}^{e}(f, 2^{-k}, \cdot)^q\Big)^{1/q}\Big| L_p(\mathbb{R}^d)\Big\|.\end{aligned}$$

In view of Proposition 1.57 we finish the proof. ■

1.3 Functions spaces defined on the unit cube

In this section we shall define the function spaces of dominating mixed smoothness on the unit cube $\Omega = [0,1]^d$. It will be convenient for us to introduce these spaces by restrictions. Let $\mathcal{D}(\Omega)$ denote the locally convex vector space of all infinitely differentiable functions with compact support in Ω. Moreover by $\mathcal{D}'(\Omega)$ we denote the set of all complex-valued distributions on Ω.

Definition 1.58. Let $0 < p, q \le \infty$ and $t \in \mathbb{R}$.
(i) Then $S^t_{p,q}B(\Omega)$ is the space of all $f \in \mathcal{D}'(\Omega)$ such that there exists a distribution $g \in S^t_{p,q}B(\mathbb{R}^d)$ satisfying $f = g|_\Omega$. It is endowed with the quotient quasi-norm

$$\| f \,|S^t_{p,q}B(\Omega)\| = \inf\left\{\|g|S^t_{p,q}B(\mathbb{R}^d)\| : \ g|_\Omega = f\right\}.$$

(ii) If $0 < p < \infty$. Then $S^t_{p,q}F(\Omega)$ is the space of all $f \in \mathcal{D}'(\Omega)$ such that there exists a distribution $g \in S^t_{p,q}F(\mathbb{R}^d)$ satisfying $f = g|_\Omega$. It is endowed with the quotient quasi-norm

$$\| f \,|S^t_{p,q}F(\Omega)\| = \inf\left\{\|g|S^t_{p,q}F(\mathbb{R}^d)\| : \ g|_\Omega = f\right\}.$$

The local version of Theorems 1.39 and 1.41 read as follows.

Theorem 1.59. *Let $t \in \mathbb{R}$ and $d > 1$.*
(i) *Let $0 < p < \infty$. Then the following formula*

$$S^t_{p,p}B(\Omega) = B^t_{p,p}([0,1]) \otimes_{\sigma_p} \ldots \otimes_{\sigma_p} B^t_{p,p}([0,1]), \qquad \textit{(d times)}$$

holds true in the sense of equivalent norms.
(ii) *Let $1 < p < \infty$. Then the following formula*

$$S^t_pH(\Omega) = H^t_p([0,1]) \otimes_{\alpha_p} \ldots \otimes_{\alpha_p} H^t_p([0,1]), \qquad \textit{(d times)}$$

holds true in the sense of equivalent norms.

The proof of Theorem 1.59 may be found in [107]. We are exclusively interested in compact embeddings of function space of dominating mixed smoothness in Chapter 4. Let us recall under which conditions the identity $S^t_{p_0,q}A(\Omega) \to L_p(\Omega)$ is compact. For the proof we refer to [140, Theorem 3.17].

Lemma 1.60. *Let $0 < p_0, q \le \infty$ (with $p_0 < \infty$ in F-case) and $1 \le p \le \infty$. Then the embedding $S^t_{p_0,q}A(\Omega) \to L_p(\Omega)$, is compact if and only if $t > (\frac{1}{p_0} - \frac{1}{p})_+$.*

We now proceed with a discussion of extension operators from $S^t_{p,q}A(\Omega)$ to $S^t_{p,q}A(\mathbb{R}^d)$. Let $0 < p < \infty$. Let $E : B^t_{p,p}([0,1]) \to B^t_{p,p}(\mathbb{R})$ denote a linear and continuous extension operator. For existence of those operators we refer, e.g., to [130, 3.3.4] or [101]. Then the d-fold tensor product operator

$$\mathcal{E}_d := E \otimes \ldots \otimes E$$

maps the tensor product space $S^t_{p,p}B(\Omega)$ into the tensor product space $S^t_{p,p}B(\mathbb{R}^d)$, see Theorems 1.39 and 1.59. Since the tensor norm σ_p is an uniform quasi-norm, it follows that $\mathcal{E}_d$ is a linear and bounded extension operator, i.e., $\mathcal{E}_d \in \mathcal{L}(S^t_{p,p}B(\Omega), S^t_{p,p}B(\mathbb{R}^d))$. A similar argument holds for Sobolev spaces $S^t_pH(\Omega)$ with $1 < p < \infty$. Extension operators on

Besov spaces $S^t_{p,p}B(\Omega)$ $(1 \le p \le \infty)$ and Sobolev spaces $S^m_pW(\mathbb{R}^d)$ have been previously considered in [133, Theorem 1.67]. For general extension operators from $S^t_{p,q}A(D_a)$ to $S^t_{p,q}A(\mathbb{R}^d)$ we refer to Ullrich [138]. Here D_a is a so-called rectangular domain, i.e.,

$$D_a = M_1 \times \ldots \times M_d,$$

where $M_i = (a_i, \infty)$ or $M_i = (-\infty, a_i)$, $i = 1, \ldots, d$ for some $a = (a_1, \ldots, a_d) \in \mathbb{R}^d$. Ullrich [138, Theorem 3.4] has proved that if $0 < p, q \le \infty$ ($p < \infty$ in F-case) and $t \in \mathbb{R}$ and D_a is a rectangular domain, then there exists a linear bounded extension operator $\mathcal{E}$ from $S^t_{p,q}A(D_a)$ into $S^t_{p,q}A(\mathbb{R}^d)$.

2 Comparison with isotropic spaces

In view of Definitions 1.12 (i) and 1.19 (i) we obtain immediately the chain of continuous embeddings
$$W_p^{md}(\mathbb{R}^d) \hookrightarrow S_p^m W(\mathbb{R}^d) \hookrightarrow W_p^m(\mathbb{R}^d)$$
with $1 < p < \infty$ and $m \in \mathbb{N}_0$. By using Fourier multipliers assertion for the spaces $L_p(\mathbb{R}^d)$ Schmeißer [102] showed that

$$H_p^{td}(\mathbb{R}^d) \hookrightarrow S_p^t H(\mathbb{R}^d) \hookrightarrow H_p^t(\mathbb{R}^d)$$

holds if $t > 0$ and $1 < p < \infty$. He also obtained the continuous embedding $S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow B^t_{p,q}(\mathbb{R}^d)$ in case of Banach spaces and positive smoothness, i.e., $t > 0$, $1 \le p, q \le \infty$. Here he employed the characterization by differences of those spaces. In this section we shall consider, under which conditions the following embeddings

$$S^t_{p,q}A(\mathbb{R}^d) \hookrightarrow A^t_{p,q}(\mathbb{R}^d) \qquad \text{and} \qquad A^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}A(\mathbb{R}^d)$$

hold true. We wish to mention that Hansen [45] also considered these types of embeddings but he used additional smoothness, i.e., he showed that

$$S^{t+\varepsilon}_{p,q_0}A(\mathbb{R}^d) \hookrightarrow A^t_{p,q}(\mathbb{R}^d) \qquad \text{and} \qquad A^{td+\varepsilon}_{p,q_0}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}A(\mathbb{R}^d)$$

with $t \in \mathbb{R}$, $0 < p \le \infty$ ($p < \infty$ in F-case), $\varepsilon > 0$ and q_0, q arbitrary.

2.1 Preparations and test functions

For us it will be convenient to switch to an equivalent quasi-norm of isotropic Besov and Triebel-Lizorkin spaces. Let $\psi_0 \in C_0^\infty(\mathbb{R}^d)$ such that

$$\psi_0(x) = 1 \text{ if } \sup_{i=1,\ldots,d} |x_i| \le 1 \qquad \text{and} \qquad \psi_0(x) = 0 \;\text{ if } \sup_{i=1,\ldots,d} |x_i| \ge \frac{3}{2}\,.$$

For $j \in \mathbb{N}$, we define $\psi_j(x) := \psi_0(2^{-j}x) - \psi_0(2^{-j+1}x)$, $x \in \mathbb{R}^d$. Then we have for $j \in \mathbb{N}$

$$\operatorname{supp} \psi_j \subset \Big\{x : \sup_{i=1,\ldots,d} |x_i| \le 3.2^{j-1}\Big\} \setminus \Big\{x : \sup_{i=1,\ldots,d} |x_i| \le 2^{j-1}\Big\}$$

and

$$\psi_j(x) = 1 \quad \text{on the set} \quad \Big\{x : \sup_{i=1,\ldots,d} |x_i| \le 2^j\Big\} \setminus \Big\{x : \sup_{i=1,\ldots,d} |x_i| \le \frac{3}{4}\, 2^j\Big\}.$$

As an easy consequence of [130, Proposition 2.3.2] one obtains the following proposition.

Proposition 2.1. *Let $t \in \mathbb{R}$ and $0 < p, q \le \infty$.*
(i) *Then the space $B^t_{p,q}(\mathbb{R}^d)$ is the collection of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that*

$$\|f|B^t_{p,q}(\mathbb{R}^d)\|^\psi = \bigg(\sum_{j=0}^\infty 2^{jtq} \big\| \mathcal{F}^{-1}(\psi_j \mathcal{F} f) | L_p(\mathbb{R}^d) \big\|^q \bigg)^{1/q}$$

is finite. The quasi-norms $\|f|B^t_{p,q}(\mathbb{R}^d)\|^\psi$ *and* $\|f|B^t_{p,q}(\mathbb{R}^d)\|^\phi$ *are equivalent.*
(ii) *Let* $0 < p < \infty$*. Then the space* $F^t_{p,q}(\mathbb{R}^d)$ *is the collection of all* $f \in \mathcal{S}'(\mathbb{R}^d)$ *such that*

$$\|f|F^t_{p,q}(\mathbb{R}^d)\|^\psi = \left\| \left(\sum_{j=0}^{\infty} 2^{jtq} |\mathcal{F}^{-1}(\psi_j \mathcal{F}f)(\cdot)|^q \right)^{1/q} \middle| L_p(\mathbb{R}^d) \right\|$$

is finite. The quasi-norms $\|f|F^t_{p,q}(\mathbb{R}^d)\|^\psi$ *and* $\|f|F^t_{p,q}(\mathbb{R}^d)\|^\phi$ *are equivalent.*

In this section we will work with the $\psi-$norm. Therefore we shall write $\|f|A^t_{p,q}(\mathbb{R}^d)\|$ instead of $\|f|A^t_{p,q}(\mathbb{R}^d)\|^\psi$. Lemma 1.6 and the homogeneity properties of the Fourier transform yield the following.

Lemma 2.2. *Let* $\{\psi_j\}_{j=0}^\infty$ *be the above system and let* $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ *be the decompositions of unity given in Remark 1.24. Let further* $0 < p \le \infty$ *and* $u = \min(1, p)$*. Then there exists a positive constant* C *such that*

$$\| \mathcal{F}^{-1}(\psi_j \varphi_k \mathcal{F}f) \,| L_p(\mathbb{R}^d)\| \le C \,\|\mathcal{F}^{-1}\varphi_k \mathcal{F}f | L_p(\mathbb{R}^d)\| \tag{2.1}$$

and

$$\|\mathcal{F}^{-1}(\varphi_k \psi_j \mathcal{F}f)|L_p(\mathbb{R}^d)\| \le C\, 2^{(jd-|k|_1)(\frac{1}{u}-1)} \|\mathcal{F}^{-1}\psi_j \mathcal{F}f|L_p(\mathbb{R}^d)\| \tag{2.2}$$

hold for all $j \in \mathbb{N}_0$*,* $k \in \mathbb{N}_0^d$ *and all* $f \in \mathcal{S}'(\mathbb{R}^d)$ *with finite right-hand sides.*

Proof. *Step 1.* Observe, that the assertion is obvious if $\psi_j \varphi_k \equiv 0$. The condition $\operatorname{supp}\psi_j \cap \operatorname{supp}\varphi_k \neq \emptyset$ implies

$$\max_{i=1,\ldots,d} k_i - 1 \le j \le \max_{i=1,\ldots,d} k_i + 1. \tag{2.3}$$

Hence, we shall prove (2.1) and (2.2) for the case $j = |k|_\infty$. The cases $j = |k|_\infty \pm 1$ can be treated in a similar way. For $k \in \mathbb{N}_0^d$ and $j \in \mathbb{N}_0$ we denote

$$\begin{aligned} \Omega_k &= \{x \in \mathbb{R}^d : |x_i| \le 2^{k_i+1},\ i = 1, \ldots, d\}, \\ \Gamma_j &= \{x \in \mathbb{R}^d : \sup_{i=1,\ldots,d} |x_i| \le 2^{j+1}\}. \end{aligned}$$

Step 2. Proof of (2.1). For $f \in \mathcal{S}'(\mathbb{R}^d)$ we put $g := \mathcal{F}^{-1}\varphi_k \mathcal{F}f$. Then we have $g \in L_p^{\Omega_k}(\mathbb{R}^d)$ and $g(2^{-j}\cdot) \in L_p^{\Gamma_0}(\mathbb{R}^d)$. Observe that

$$\begin{aligned} \|\mathcal{F}^{-1}\psi_j \mathcal{F}g|L_p(\mathbb{R}^d)\| &= 2^{-\frac{jd}{p}} \|(\mathcal{F}^{-1}\psi_j \mathcal{F}g)(2^{-j}\cdot)|L_p(\mathbb{R}^d)\| \\ &= 2^{-\frac{jd}{p}} \|\mathcal{F}^{-1}\big(\psi_j(2^j\cdot)\mathcal{F}[g(2^{-j}\cdot)]\big)|L_p(\mathbb{R}^d)\|. \end{aligned}$$

We assume that $j > 1$. Lemma 1.6 together with $\operatorname{supp}\psi_j(2^j\cdot) \subset \Gamma_0$ yield

$$\begin{aligned} \|\mathcal{F}^{-1}\psi_j \mathcal{F}g|L_p(\mathbb{R}^d)\| &\le c_1\, 2^{-\frac{jd}{p}} \|\mathcal{F}^{-1}(\psi_1(2\,\cdot))\,|L_u(\mathbb{R}^d)\| \cdot \|g(2^{-j}\cdot)|L_p(\mathbb{R}^d)\| \\ &\le c_2\, \|\mathcal{F}^{-1}\psi_0\,|L_u(\mathbb{R}^d)\| \cdot \|g|L_p(\mathbb{R}^d)\|. \end{aligned}$$

A similar argument yields the estimate of $\mathcal{F}^{-1}\psi_0 \mathcal{F}g$. This proves (2.1).
Step 3. To prove (2.2), we put $h := \mathcal{F}^{-1}\psi_j \mathcal{F}f$. Then we have $h \in L_p^{\Gamma_j}(\mathbb{R}^d)$, hence

$h(2^{-j}\cdot) \in L_p^{\Gamma_0}(\mathbb{R}^d)$. In addition we know that $\operatorname{supp}\varphi_k(2^j\cdot) \subset \Gamma_0$ if $\psi_0 \cdot \varphi_k \not\equiv 0$. If $k \in \mathbb{N}_0^d$ such that $k \geq \bar{1}$ then from Lemma 1.6 we obtain

$$\begin{aligned}\|\mathcal{F}^{-1}\varphi_k\mathcal{F}h|L_p(\mathbb{R}^d)\| &= 2^{-\frac{jd}{p}}\|(\mathcal{F}^{-1}\varphi_k\mathcal{F}h)(2^{-j}\cdot)|L_p(\mathbb{R}^d)\| \\ &= 2^{-\frac{jd}{p}}\|\mathcal{F}^{-1}\big[\varphi_k(2^j\cdot)\mathcal{F}[h(2^{-j}\cdot)]\big]|L_p(\mathbb{R}^d)\| \\ &\leq c_1\, 2^{-\frac{jd}{p}}\|\mathcal{F}^{-1}[\varphi_k(2^j\cdot)]|L_u(\mathbb{R}^d)\|\cdot\|h(2^{-j}\cdot)|L_p(\mathbb{R}^d)\| \\ &\leq c_1\,\|\mathcal{F}^{-1}[\varphi_k(2^j\cdot)]|L_u(\mathbb{R}^d)\|\cdot\|h|L_p(\mathbb{R}^d)\|.\end{aligned} \tag{2.4}$$

The homogeneity properties of the Fourier transform lead to

$$\begin{aligned}\|\mathcal{F}^{-1}[\varphi_k(2^j\cdot)]|L_u(\mathbb{R}^d)\| &= \|\mathcal{F}^{-1}[\varphi_{\bar{1}}(2^{-k+\bar{j}+\bar{1}}\diamond\cdot)]|L_u(\mathbb{R}^d)\| \\ &\leq c_2\, 2^{(jd-|k|_1)(\frac{1}{u}-1)}\,\|\mathcal{F}^{-1}\varphi_{\bar{1}}|L_u(\mathbb{R}^d)\|.\end{aligned}$$

Inserting this into (2.4) we get (2.2) for those k. An obvious modification yields the estimate for the remaining k. The proof is complete. ■

We proceed with the investigation of some test functions. These functions will be used to prove the optimality of the above-mentioned embeddings. Recall that $0 < p < \infty$ in the F- case.

Example 1

Let us fix $\ell \in \mathbb{N}$ and let $\eta \in \mathcal{S}(\mathbb{R})$ such that $\operatorname{supp}(\mathcal{F}\eta) \subset \{\xi \in \mathbb{R} : 0 < \xi < \frac{1}{4}\}$. We define the function g_ℓ by its Fourier transform

$$\mathcal{F}g_\ell(\xi) = \sum_{j=1}^{\ell} a_j(\mathcal{F}\eta)(\xi - \frac{7}{8}2^j), \qquad a_j \in \mathbb{C}\,,\ j = 1,\ldots,\ell.$$

Then we arrive at

$$\mathcal{F}^{-1}\big(\varphi_j\mathcal{F}g_\ell\big)(\xi) = a_j e^{\frac{7}{8}2^j i\xi}\eta(\xi)\,, \qquad j \leq \ell,$$

where φ_j given in Remark 1.24. Consequently we obtain

$$\|g_\ell|B^0_{p,q}(\mathbb{R})\| = \|g_\ell|F^0_{p,q}(\mathbb{R})\| = \|\eta|L_p(\mathbb{R})\|\Big(\sum_{j=1}^{\ell}|a_j|^q\Big)^{1/q}. \tag{2.5}$$

Now we turn to the multi-dimension by introduce a family of test functions $f_\ell : \mathbb{R}^d \to \mathbb{C}$ as follows

$$\mathcal{F}f_\ell(x) = \theta_\ell(x_1)\cdot\ldots\cdot\theta_\ell(x_{d-1})(\mathcal{F}g_\ell)(x_d)\,, \qquad x = (x_1,\ldots,x_d) \in \mathbb{R}^d\,,$$

where $\theta \in \mathcal{S}(\mathbb{R})$ is a function satisfying

$$\operatorname{supp}\theta \subset \{\xi \in \mathbb{R} :\ 3/4 \leq |\xi| \leq 1\} \qquad \text{and} \qquad \theta_\ell(\xi) = \theta(2^{-\ell}\xi).$$

Clearly,

$$\operatorname{supp}\theta_\ell \subset \{\xi \in \mathbb{R} :\ \varphi_\ell(\xi) = 1\} \qquad \text{and} \qquad \operatorname{supp}(\mathcal{F}f_\ell) \subset \{x \in \mathbb{R}^d :\ \psi_\ell(x) = 1\}.$$

By mean of cross-quasi-norm property we obtain

$$\begin{aligned}
\|f_\ell|S^0_{p,q}B(\mathbb{R}^d)\| &= \|f_\ell|S^0_{p,q}F(\mathbb{R}^d)\| = \|\mathcal{F}^{-1}\theta_\ell\,|F^0_{p,q}(\mathbb{R})\|^{d-1}\cdot\|g_\ell|F^0_{p,q}(\mathbb{R})\| \\
&= \|\mathcal{F}^{-1}\theta_\ell\,|L_p(\mathbb{R})\|^{d-1}\cdot\|g_\ell|F^0_{p,q}(\mathbb{R})\| \\
&= 2^{\ell(1-\frac{1}{p})(d-1)}\,\|\,\mathcal{F}^{-1}\theta\,|L_p(\mathbb{R}^d)\|^{d-1}\cdot\|g_\ell|F^0_{p,q}(\mathbb{R})\| \\
&= C_1\,2^{\ell(1-\frac{1}{p})(d-1)}\Big(\sum_{j=1}^{\ell}|a_j|^q\Big)^{1/q}
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\|f_\ell|B^0_{p,q}(\mathbb{R}^d)\| = \|f_\ell|F^0_{p,q}(\mathbb{R}^d)\| &= \|\mathcal{F}^{-1}\theta_\ell\,|L_p(\mathbb{R})\|^{d-1}\cdot\|g_\ell|L_p(\mathbb{R})\| \\
&= C_1\,2^{\ell(1-\frac{1}{p})(d-1)}\|g_\ell|L_p(\mathbb{R})\|
\end{aligned}$$

with an appropriate positive constant C_1. In case $1 < p < \infty$, using Littlewood-Paley characterization of $L_p(\mathbb{R})$ we have

$$\begin{aligned}
\|f_\ell|B^0_{p,q}(\mathbb{R}^d)\| = \|f_\ell|F^0_{p,q}(\mathbb{R}^d)\| &\asymp C_1\,2^{\ell(1-\frac{1}{p})(d-1)}\|g_\ell|F^0_{p,2}(\mathbb{R})\| \\
&= C_2\,2^{\ell(1-\frac{1}{p})(d-1)}\Big(\sum_{j=1}^{\ell}|a_j|^2\Big)^{1/2}.
\end{aligned} \tag{2.7}$$

Example 2

In case $p=\infty$ nontrivial periodic functions are contained in $B^t_{\infty,q}(\mathbb{R}^d)$ and $S^t_{\infty,q}B(\mathbb{R}^d)$. So we can work directly with lacunary series. Let

$$f_\ell(x) := \sum_{j=1}^{\ell} a_j\, e^{i(2^\ell x_1 + 2^j x_2)}\,, \qquad x=(x_1,\ldots,x_d)\in\mathbb{R}^d\,.$$

Then

$$\mathcal{F}^{-1}(\psi_m\mathcal{F}f_\ell) = \delta_{m,\ell}\, f_\ell$$

and

$$\mathcal{F}^{-1}(\varphi_k\mathcal{F}f_\ell)(x) = \delta_{k,(\ell,j,0,\ldots,0)}\, a_j\, e^{i(2^\ell x_1+2^j x_2)}$$

follow. For $a_j\geq 0$ for all j this will allow us to calculate the quasi-norms in $B^t_{\infty,q}(\mathbb{R}^d)$ and $S^t_{\infty,q}B(\mathbb{R}^d)$. We obtain in the first case

$$\begin{aligned}
\|\,f_\ell\,|B^t_{\infty,q}(\mathbb{R}^d)\| &= 2^{\ell t}\,\big\|\mathcal{F}^{-1}(\psi_\ell\mathcal{F}f)(\cdot)\big|L_\infty(\mathbb{R}^d)\big\| \\
&= 2^{\ell t}\sup_{x\in\mathbb{R}^d}\Big|\sum_{j=1}^{\ell}a_j\,e^{i(2^\ell x_1+2^j x_2)}\Big| = 2^{\ell t}\sum_{j=1}^{\ell}a_j\,.
\end{aligned} \tag{2.8}$$

Concerning the dominating mixed smoothness we conclude

$$\begin{aligned}
\|\,f_\ell\,|S^t_{\infty,q}B(\mathbb{R}^d)\| &= \Big(\sum_{j=1}^{\ell}2^{(j+\ell)tq}\,\big\|\mathcal{F}^{-1}(\varphi_{(\ell,j,0,\ldots,0)}\mathcal{F}f_\ell)(\cdot)\,\big|L_\infty(\mathbb{R}^d)\big\|^q\Big)^{1/q} \\
&= 2^{\ell t}\Big(\sum_{j=1}^{\ell}2^{jtq}\,|a_j|^q\Big)^{1/q}.
\end{aligned} \tag{2.9}$$

Example 3

Let us consider a function $g \in C_0^\infty(\mathbb{R})$ such that $\operatorname{supp} g \subset \{\xi \in \mathbb{R} : 3/4 \le |\xi| \le 1\}$. For $j \in \mathbb{N}_0$, $k \in \mathbb{N}_0^d$ we define $g_j(\xi) = g(2^{-j}\xi)$, $\xi \in \mathbb{R}$, and

$$g_k(x) = g_{k_1}(x_1) \cdot \ldots \cdot g_{k_d}(x_d) , \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d .$$

For $\ell \in \mathbb{N}$ we put

$$\nabla_\ell := \Big\{ k \in \mathbb{N}_0^d,\ |k|_\infty = \ell \Big\} .$$

Then, if $k \in \nabla_\ell$, we have

$$\operatorname{supp}(g_k) \subset \Big\{ x \in \mathbb{R}^d :\ \varphi_k(x) = 1 \Big\} \subset \Big\{ x \in \mathbb{R}^d :\ \psi_\ell(x) = 1 \Big\}.$$

We define the family of test functions

$$f_\ell = \sum_{k \in \nabla_\ell} a_k \, \mathcal{F}^{-1} g_k .$$

The coefficients $\{a_k\}_{k \in \nabla_\ell}$ will be chosen later on. By construction we have

$$\begin{aligned} \| f_\ell | S^0_{p,q} B(\mathbb{R}^d) \| &= \bigg(\sum_{k \in \mathbb{N}_0^d} \big\| \mathcal{F}^{-1}(\varphi_k \mathcal{F} f_\ell) \big| L_p(\mathbb{R}^d) \big\|^q \bigg)^{1/q} \\ &= \bigg(\sum_{k \in \nabla_\ell} |a_k|^q \big\| \mathcal{F}^{-1} g_k \big| L_p(\mathbb{R}^d) \big\|^q \bigg)^{1/q} . \end{aligned}$$

Observe that

$$\big\| \mathcal{F}^{-1} g_k \big| L_p(\mathbb{R}^d) \big\| = \prod_{i=1}^d \big\| \mathcal{F}^{-1} g_{k_i} \big| L_p(\mathbb{R}) \big\| = 2^{|k|_1 (1 - \frac{1}{p})} \, \big\| \, \mathcal{F}^{-1} g \big| L_p(\mathbb{R}) \big\|^d = C \, 2^{|k|_1 (1-\frac{1}{p})} .$$

Here $C = \| \, \mathcal{F}^{-1} g | L_p(\mathbb{R}) \|^d$. Consequently we obtain

$$\| f_\ell | S^0_{p,q} B(\mathbb{R}^d) \| \ = \ C \bigg(\sum_{k \in \nabla_\ell} |a_k|^q \, 2^{|k|_1 (1 - \frac{1}{p}) q} \bigg)^{1/q} , \qquad \ell \in \mathbb{N} .$$

Next we compute

$$\begin{aligned} \| f_\ell | B^0_{p,q}(\mathbb{R}^d) \| &= \bigg(\sum_{j=0}^\infty \big\| \mathcal{F}^{-1}(\psi_j \mathcal{F} f_\ell) \big| L_p(\mathbb{R}^d) \big\|^q \bigg) \\ &= \bigg\| \sum_{k \in \nabla_\ell} a_k \mathcal{F}^{-1} g_k \bigg| L_p(\mathbb{R}^d) \bigg\|. \end{aligned}$$

Recall, for $0 < p_0 < p < p_1 < \infty$ we have

$$S^{\frac{1}{p_0} - \frac{1}{p}}_{p_0,p} B(\mathbb{R}^d) \hookrightarrow S^0_{p,2} F(\mathbb{R}^d) \hookrightarrow S^{\frac{1}{p_1} - \frac{1}{p}}_{p_1,p} B(\mathbb{R}^d) ,$$

see Lemma 1.34 and $S^0_{p,2}F(\mathbb{R}^d) = L_p(\mathbb{R}^d)$, $1 < p < \infty$, see Theorem 1.27. These arguments lead to

$$\begin{aligned}\Big\| \sum_{k\in\nabla_\ell} a_k \mathcal{F}^{-1} g_k \Big| L_p(\mathbb{R}^d)\Big\| &\le C_1 \Big(\sum_{k\in\nabla_\ell} 2^{|k|_1(\frac{1}{p_0}-\frac{1}{p})p} |a_k|^p \|\mathcal{F}^{-1} g_k | L_{p_0}(\mathbb{R}^d)\|^p \Big)^{1/p} \\ &= C_2 \Big(\sum_{k\in\nabla_\ell} 2^{|k|_1(\frac{1}{p_0}-\frac{1}{p})p} |a_k|^p 2^{|k|_1(1-\frac{1}{p_0})p} \Big)^{1/p} \\ &= C_2 \Big(\sum_{k\in\nabla_\ell} |a_k|^p 2^{|k|_1(1-\frac{1}{p})p} \Big)^{1/p} .\end{aligned}$$

Similarly we have

$$\begin{aligned}\Big(\sum_{k\in\nabla_\ell} 2^{|k|_1(\frac{1}{p_1}-\frac{1}{p})p} |a_k|^p \|\mathcal{F}^{-1} g_k | L_{p_1}(\mathbb{R}^d)\|^p \Big)^{1/p} &= C_3 \Big(\sum_{k\in\nabla_\ell} |a_k|^p 2^{|k|_1(1-\frac{1}{p})p} \Big)^{1/p} \\ &\le C_4 \Big\| \sum_{k\in\nabla_\ell} a_k \mathcal{F}^{-1} g_k \Big| L_p(\mathbb{R}^d)\Big\| .\end{aligned}$$

Altogether we have proved in case $1 < p < \infty$

$$\|f_\ell | B^0_{p,q}(\mathbb{R}^d)\| \asymp \Big(\sum_{k\in\nabla_\ell} |a_k|^p 2^{|k|_1(1-\frac{1}{p})p} \Big)^{1/p} ,$$

where the positive constants behind $\asymp$ do not depend on $\ell \in \mathbb{N}$.

Example 4

We consider the same functions g_k, $k \in \mathbb{N}_0^d$, as in Example 3. For $\ell \in \mathbb{N}_0$ we define

$$f_\ell := \mathcal{F}^{-1} g_{\bar{\ell}_*} \qquad \bar{\ell}_* := (\ell, 0, \ldots, 0) \in \mathbb{N}_0^d .$$

Then we find

$$\|f_\ell | B^t_{p,q}(\mathbb{R}^d)\| = \|f_\ell | F^t_{p,q}(\mathbb{R}^d)\| = 2^{t\ell} \big\| \mathcal{F}^{-1} g_{\bar{\ell}_*} \big| L_p(\mathbb{R}^d)\big\| = C\, 2^{\ell(t+1-\frac{1}{p})} \tag{2.10}$$

and

$$\|f_\ell | S^t_{p,q}B(\mathbb{R}^d)\| = \|f_\ell | S^t_{p,q}F(\mathbb{R}^d)\| = 2^{t\ell} \big\| \mathcal{F}^{-1} g_{\bar{\ell}_*} \big| L_p(\mathbb{R}^d)\big\| = C\, 2^{\ell(t+1-\frac{1}{p})} . \tag{2.11}$$

Here $C = \|\mathcal{F}^{-1} g | L_p(\mathbb{R})\|^d$ as in Example 3. If we put $h_\ell := \sum_{j=0}^{\ell} a_j f_j$ then we obtain

$$\|h_\ell | B^t_{p,q}(\mathbb{R}^d)\| = \|h_\ell | S^t_{p,q}B(\mathbb{R}^d)\| = C \Big(\sum_{j=0}^{\ell} 2^{j(t+1-\frac{1}{p})q} |a_j|^q \Big)^{1/q} . \tag{2.12}$$

Example 5

We consider the same functions g_k, $k \in \mathbb{N}_0^d$, as in Example 3. This time we define

$$f_\ell := \mathcal{F}^{-1} g_{\bar{\ell}}\,, \qquad \text{and} \qquad h_\ell := \sum_{j=0}^{\ell} a_j f_j\,, \quad \bar{\ell} = (\ell, \ldots, \ell) \in \mathbb{N}_0^d.$$

As above we conclude

$$\|f_\ell | S^t_{p,q}B(\mathbb{R}^d)\| = \|f_\ell | S^t_{p,q}F(\mathbb{R}^d)\| = 2^{td\ell} \big\| \mathcal{F}^{-1} g_{\bar{\ell}} \big| L_p(\mathbb{R}^d)\big\| = C\, 2^{d\ell(t+1-\frac{1}{p})} \tag{2.13}$$

and

$$\|f_\ell | B^t_{p,q}(\mathbb{R}^d)\| = \|f_\ell | F^t_{p,q}(\mathbb{R}^d)\| = 2^{t\ell} \big\| \mathcal{F}^{-1} g_{\bar{\ell}} \big| L_p(\mathbb{R}^d)\big\| = C\, 2^{d\ell(\frac{t}{d}+1-\frac{1}{p})} \tag{2.14}$$

The constant C here is the same as in Example 3. Concerning the norm of the function $h_\ell = \sum_{j=0}^{\ell} a_j f_j$ we obtain

$$\|h_\ell | S^t_{p,q}B(\mathbb{R}^d)\| = C \bigg(\sum_{j=0}^{\ell} 2^{jd(t+1-\frac{1}{p})q} |a_j|^q \bigg)^{1/q} \tag{2.15}$$

and

$$\|h_\ell | B^t_{p,q}(\mathbb{R}^d)\| = C \bigg(\sum_{j=0}^{\ell} 2^{jd(\frac{t}{d}+1-\frac{1}{p})q} |a_j|^q \bigg)^{1/q}. \tag{2.16}$$

Example 6

Let $g \in \mathcal{S}(\mathbb{R}^d)$ with $\operatorname{supp}(\mathcal{F}g) \subset [0, \frac{1}{4}]^d$. We define

$$f_\ell(x) = \sum_{j=1}^{\ell} a_j e^{i\frac{7}{8}2^j x_1} g(x)\,, \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d.$$

Then we have

$$\mathcal{F} f_\ell(x) = \sum_{j=1}^{\ell} a_j (\mathcal{F}g)(x_1 - \frac{7}{8}2^j, x_2, \ldots, x_d)\,.$$

We obtain

$$\mathcal{F}^{-1} \varphi_k \mathcal{F} f_\ell(x) = \sum_{j=1}^{\ell} \delta_{k,(j,0,\ldots,0)} a_j e^{i\frac{7}{8}2^j x_1} g(x)$$

and

$$\mathcal{F}^{-1} \psi_j \mathcal{F} f_\ell(x) = a_j e^{i\frac{7}{8}2^j x_1} g(x)\,, \qquad j \le \ell\,.$$

This leads to

$$\|f_\ell | F^t_{p,q}(\mathbb{R}^d)\| = \|f_\ell | S^t_{p,q}F(\mathbb{R}^d)\| = \|g | L_p(\mathbb{R}^d)\| \Big(\sum_{j=1}^{\ell} 2^{jtq} |a_j|^q \Big)^{1/q}.$$

Example 7

We shall modify Example 6. This time we define the function

$$f_\ell(x) = \sum_{j=1}^{\ell} a_j e^{i\frac{7}{8}2^j(x_1+\ldots+x_d)} g(x)\,, \qquad x=(x_1,\ldots,x_d)\in\mathbb{R}^d.$$

As above we conclude

$$\mathcal{F}^{-1}\varphi_k\mathcal{F} f_\ell(x) = \sum_{j=1}^{\ell} \delta_{k,\bar{j}} a_j e^{i\frac{7}{8}2^j(x_1+\ldots+x_d)} g(x)$$

and

$$\mathcal{F}^{-1}\psi_j\mathcal{F} f_\ell(x) = a_j e^{i\frac{7}{8}2^j(x_1+\ldots+x_d)} g(x)\,, \qquad j\le\ell,$$

here $\bar{j}=(j,\ldots,j)\in\mathbb{N}^d$. Hence, we obtain

$$\|f_\ell|F^t_{p,q}(\mathbb{R}^d)\| = \|g|L_p(\mathbb{R}^d)\|\Big(\sum_{j=1}^{\ell} 2^{jtq}|a_j|^q\Big)^{1/q}$$

and

$$\|f_\ell|S^t_{p,q}F(\mathbb{R}^d)\| = \|g|L_p(\mathbb{R}^d)\|\Big(\sum_{j=1}^{\ell} 2^{djtq}|a_j|^q\Big)^{1/q}.$$

Example 8

This example is taken from [130, Section 2.3.9]. Let $g\in\mathcal{S}(\mathbb{R}^d)$ be a function such that $\operatorname{supp}(\mathcal{F}g)\subset\{x\in\mathbb{R}^d:\ |x|\le 1\}$. We define

$$h_\ell(x) := g(2^{-\ell}x)\,, \qquad x\in\mathbb{R}^d\,, \quad \ell\in\mathbb{N}\,.$$

For all p,q,t we conclude

$$\|\,h_\ell\,|B^t_{p,q}(\mathbb{R}^d)\| = \|\,h_\ell\,|F^t_{p,q}(\mathbb{R}^d)\| = \|\,h_\ell\,|L_p(\mathbb{R}^d)\| = 2^{\ell d/p}\,\|\,g\,|L_p(\mathbb{R}^d)\|\,, \qquad \ell\in\mathbb{N}\,.$$

Similarly, also for all p,q,t, we obtain

$$\|\,h_\ell\,|S^t_{p,q}B(\mathbb{R}^d)\| = \|\,h_\ell\,|S^t_{p,q}F(\mathbb{R}^d)\| = \|\,h_\ell\,|L_p(\mathbb{R}^d)\| = 2^{\ell d/p}\,\|\,g\,|L_p(\mathbb{R}^d)\|\,, \qquad \ell\in\mathbb{N}\,.$$

As an immediate consequence of these two identities we get the following.

Lemma 2.3. *Let $0<p_0,p_1,q_0,q_1\le\infty$ ($p_0,p_1<\infty$ in F-case) and $t_0,t_1\in\mathbb{R}$.*
(i) *An embedding $S^{t_0}_{p_0,q_0}A(\mathbb{R}^d)\hookrightarrow A^{t_1}_{p_1,q_1}(\mathbb{R}^d)$ implies $p_0\le p_1$.*
(ii) *An embedding $A^{t_0}_{p_0,q_0}(\mathbb{R}^d)\hookrightarrow S^{t_1}_{p_1,q_1}A(\mathbb{R}^d)$ implies $p_0\le p_1$.*

2.2 The case of Besov spaces

2.2.1 The embedding of dominating mixed spaces into isotropic spaces

Theorem 2.4. *Let $d \geq 2$, $0 < p, q \leq \infty$ and $t \in \mathbb{R}$. Then we have*

$$S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow B^t_{p,q}(\mathbb{R}^d) \tag{2.17}$$

if and only if one of the following conditions is satisfied

- $t > 0$;
- $t = 0$, $0 < p < \infty$ *and* $0 < q \leq \min(p, 2)$;
- $t = 0$, $p = \infty$ *and* $q \leq 1$.

We need the following lemma.

Lemma 2.5. (i) *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Then the embedding*

$$S^0_{p,q}B(\mathbb{R}^d) \hookrightarrow B^0_{p,q}(\mathbb{R}^d) \qquad \textit{implies} \qquad q \leq \min(2, p).$$

(ii) *Let $0 < q \leq \infty$. Then the embedding*

$$S^0_{\infty,q}B(\mathbb{R}^d) \hookrightarrow B^0_{\infty,q}(\mathbb{R}^d) \qquad \textit{implies} \qquad q \leq 1.$$

Proof. *Step 1.* We prove (i).
Substep 1.1. We show necessity of $q \leq 2$. Temporarily we assume $1 < p < \infty$. We use our test functions from Example 1. The embedding $S^0_{p,q}B(\mathbb{R}^d) \hookrightarrow B^0_{p,q}(\mathbb{R}^d)$ implies the existence of a constant $c > 0$ such that

$$2^{\ell(1-\frac{1}{p})(d-1)} \Big(\sum_{j=1}^{\ell} |a_j|^2 \Big)^{1/2} \leq c\, 2^{\ell(1-\frac{1}{p})(d-1)} \Big(\sum_{j=1}^{\ell} |a_j|^q \Big)^{1/q}$$

where c does not depend on ℓ and $\{a_j\}_j$, see (2.6) and (2.7). This requires $q \leq 2$. Now we turn to $0 < p \leq 1$. Again we shall work with Example 1. For any such p there exists some real number $\Theta \in (0, 1)$ such that $\frac{2}{3} = \frac{1-\Theta}{p} + \frac{\Theta}{2}$. Lyapunov's inequality

$$\| \, g_\ell \, | L_{3/2}(\mathbb{R}) \| \leq \| \, g_\ell \, | L_p(\mathbb{R}) \|^{1-\Theta} \, \| \, g_\ell \, | L_2(\mathbb{R}) \|^{\Theta}$$

and Littlewood-Paley characterization of $L_p(\mathbb{R})$ lead us to

$$\Big(\sum_{j=1}^{\ell} |a_j|^2 \Big)^{1/2} \leq c \| \, g_\ell \, | L_p(\mathbb{R}) \|$$

with c independent of ℓ and $\{a_j\}_j$, see (2.5). Hence we obtain

$$2^{\ell(1-\frac{1}{p})(d-1)} \Big(\sum_{j=1}^{\ell} |a_j|^2 \Big)^{1/2} \leq c 2^{\ell(1-\frac{1}{p})(d-1)} \| \, g_\ell \, | L_p(\mathbb{R}) \| \asymp \| f_\ell | B^0_{p,q}(\mathbb{R}^d) \| \, .$$

Taking into account (2.6) we can argue as in case $1 < p < \infty$.
Substep 1.2. We show necessity of $q \leq p$. Therefore we use Example 3. In case $1 < p < \infty$

we choose $a_k = 2^{|k|_1(\frac{1}{p}-1)}$, $k \in \nabla_\ell$. Then we can conclude $q \le p$. Now we assume there exist $0 < p \le 1$ and $p < q \le 2$ such that

$$S^0_{p,q}B(\mathbb{R}^d) \hookrightarrow B^0_{p,q}(\mathbb{R}^d)\,.$$

In this situation we may choose a triple (p_1, q_1, Θ) such that

$$1 < p_1 < q_1 \le 2, \quad \Theta \in (0,1), \quad \frac{1}{p_1} = \frac{\Theta}{p} + \frac{1-\Theta}{2} \quad \text{and} \quad \frac{1}{q_1} = \frac{\Theta}{q} + \frac{1-\Theta}{2}.$$

Then it follows from Propositions 1.49 and 1.51 that

$$S^0_{p_1,q_1}B(\mathbb{R}^d) = [S^0_{p,q}B(\mathbb{R}^d), S^0_{2,2}B(\mathbb{R}^d)]_\Theta \quad \text{and} \quad B^0_{p_1,q_1}(\mathbb{R}^d) = [B^0_{p,q}(\mathbb{R}^d), B^0_{2,2}(\mathbb{R}^d)]_\Theta.$$

Now Proposition 1.48 yields $S^0_{p_1,q_1}B(\mathbb{R}^d) \hookrightarrow B^0_{p_1,q_1}(\mathbb{R}^d)$. But this is a contradiction to Example 3.
Step 2. To prove (ii) we use test functions from Example 2. The embedding $S^0_{\infty,q}B(\mathbb{R}^d) \hookrightarrow B^0_{\infty,q}(\mathbb{R}^d)$ implies the existence of a constant c such that

$$\sum_{j=1}^{\ell} |a_j| = \|\, f_\ell\,|B^0_{\infty,q}(\mathbb{R}^d)\| \le c\,\|\, f_\ell\,|S^0_{\infty,q}B(\mathbb{R}^d)\| \asymp \Big(\sum_{j=1}^{\ell} |a_j|^q\Big)^{1/q}$$

where c does not depend on ℓ and $\{a_j\}_j$, see (2.8) and (2.9). Choosing $a_j = 1$, $j = 1, \ldots, \ell$, it is obvious that this can happen only if $q \le 1$. ■

Proof of Theorem 2.4. We divide the proof into two parts.
Part I - sufficiency. *Step 1.* For $k \in \mathbb{N}_0^d$ we define

$$\Box_k := \{j \in \mathbb{N}_0 : \ \operatorname{supp} \psi_j \cap \operatorname{supp} \varphi_k \neq \emptyset\}$$

and $j \in \mathbb{N}_0$

$$\Delta_j := \{k \in \mathbb{N}_0^d : \ \operatorname{supp} \psi_j \cap \operatorname{supp} \varphi_k \neq \emptyset\}.$$

Recall that the condition $\operatorname{supp} \psi_j \cap \operatorname{supp} \varphi_k \neq \emptyset$ implies $|k|_\infty - 1 \le j \le |k|_\infty + 1$, see (2.3). Consequently we obtain

$$|\Box_k| \asymp 1\,, \quad k \in \mathbb{N}_0^d \qquad \text{and} \qquad |\Delta_j| \asymp (1+j)^{d-1}\,, \quad j \in \mathbb{N}_0\,. \tag{2.18}$$

By definition we have

$$\psi_j(x) = \sum_{k \in \Delta_j} \varphi_k(x)\psi_j(x)\,, \qquad x \in \mathbb{R}^d\,. \tag{2.19}$$

Step 2. Let $t > 0$ and $u = \min(1,p)$. Employing (2.19) we find

$$\begin{aligned}\|f|B^t_{p,q}(\mathbb{R}^d)\|^q &= \sum_{j=0}^{\infty} 2^{jtq} \Big\| \sum_{k \in \Delta_j} \mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f \Big| L_p(\mathbb{R}^d)\Big\|^q \\ &\le \sum_{j=0}^{\infty} 2^{jtq} \Big(\sum_{k \in \Delta_j} \|\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|^u\Big)^{q/u}.\end{aligned}$$

Using (2.1) it follows

$$\|f|B^t_{p,q}(\mathbb{R}^d)\|^q \;\le\; c_1 \sum_{j=0}^{\infty} \bigg(\sum_{k\in\Delta_j} \Big[2^{(j-|k|_1)t} 2^{|k|_1 t} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|\Big]^u \bigg)^{q/u}. \tag{2.20}$$

If $\frac{q}{u}\le 1$ we have

$$\begin{aligned}\|f|B^t_{p,q}(\mathbb{R}^d)\|^q &\le c_1 \sum_{j=0}^{\infty} \sum_{k\in\Delta_j} 2^{(j-|k|_1)tq} 2^{|k|_1 tq} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q \\ &\le c_2 \sum_{k\in\mathbb{N}_0^d} \sum_{j\in\square_k} 2^{|k|_1 tq} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q.\end{aligned} \tag{2.21}$$

The last inequality is due to $2^{(j-|k|_1)tq}\le C$ since $t>0$ and $|k|_1\ge j-1$, see (2.3). In the case $\frac{q}{u}>1$ we use Hölder's inequality with $1=\frac{u}{q}+(1-\frac{u}{q})$. From (2.20) we obtain

$$\|f|B^t_{p,q}(\mathbb{R}^d)\|^q \;\le\; c_1 \sum_{j=0}^{\infty} \sum_{k\in\Delta_j} 2^{|k|_1 tq} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q \bigg(\sum_{k\in\Delta_j} \Big[2^{(j-|k|_1)t}\Big]^{qu/(q-u)} \bigg)^{(q-u)/u}.$$

Observe, for $t>0$ we have

$$\sup_{j\in\mathbb{N}_0} \bigg(\sum_{k\in\Delta_j} \Big[2^{(j-|k|_1)t}\Big]^{qu/(q-u)} \bigg)^{(q-u)/u} < \infty\,, \tag{2.22}$$

see (2.3). Hence

$$\|f|B^t_{p,q}(\mathbb{R}^d)\|^q \;\le\; c_3 \sum_{k\in\mathbb{N}_0^d} \sum_{j\in\square_k} 2^{|k|_1 tq} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q. \tag{2.23}$$

Finally, from (2.21), (2.23) together with $\square_k \asymp 1$ we conclude

$$\|f|B^t_{p,q}(\mathbb{R}^d)\|^q \;\le\; c_4 \sum_{k\in\mathbb{N}_0^d} 2^{|k|_1 tq} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q.$$

This proves (2.17).
Step 3. Let $t=0$.
Substep 3.1. First we assume that $q\le\min(p,1)$. From (2.20) with $t=0$ we have

$$\begin{aligned}\|f|B^0_{p,q}(\mathbb{R}^d)\|^q &\le c_1 \sum_{j=0}^{\infty} \sum_{k\in\Delta_j} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q \\ &= c_1 \sum_{k\in\mathbb{N}_0^d} \sum_{j\in\square_k} \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f|L_p(\mathbb{R}^d)\|^q.\end{aligned}$$

Since $\square_k \asymp 1$ we obtain

$$\|f|B^0_{p,q}(\mathbb{R}^d)\|^q \quad\le\quad c_2\, \|f|S^0_{p,q}B(\mathbb{R}^d)\|^q\,.$$

Substep 3.2. Let $1 < p < \infty$ and $0 < q \le \min(2,p)$. Our main tool will be the Littlewood-Paley assertion. We proceed as in Step 2. Employing (2.19) and (1.9) with $t = 0$ and f replaced by $\mathcal{F}^{-1}\psi_j\mathcal{F}f$ we find

$$\begin{aligned}\|f|B^0_{p,q}(\mathbb{R}^d)\|^q &= \sum_{j=0}^{\infty} \|\mathcal{F}^{-1}\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|^q \\ &\le c_5 \sum_{j=0}^{\infty} \Big\|\Big(\sum_{k\in\Delta_j} |\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f|^2\Big)^{1/2}\Big|L_p(\mathbb{R}^d)\Big\|^q.\end{aligned}$$

Because of $\|\cdot|L_p(\ell_2)\| \le \|\cdot|\ell_{\min(2,p)}(L_p)\| \le \|\cdot|\ell_q(L_p)\|$ we deduce

$$\begin{aligned}\|f|B^0_{p,q}(\mathbb{R}^d)\|^q &\le c_5 \sum_{j=0}^{\infty}\sum_{k\in\Delta_j} \|\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|^q \\ &\le c_6 \sum_{j=0}^{\infty}\sum_{k\in\Delta_j} \|\mathcal{F}^{-1}\varphi_k\mathcal{F}f|L_p(\mathbb{R}^d)\|^q\end{aligned}$$

where we used (2.1) in the last step. As in Step 2 we can continue the estimate by changing the order of summation and using $|\square_k| \asymp 1$.
Part II - necessity. *Step 1.* Let $t = 0$. Then the necessity of $q \le \min(p,2)$ if $p < \infty$ and of $q \le 1$ if $p = \infty$ follows from Lemma 2.5.
Step 2. The case $t < 0$. We employ the test functions from Example 5. Assume that $S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow B^t_{p,q}(\mathbb{R}^d)$. From (2.13) and (2.14) we have

$$2^{d\ell(\frac{t}{d}+1-\frac{1}{p})} \le c2^{d\ell(t+1-\frac{1}{p})} \iff 2^{t\ell} \le c2^{dt\ell}$$

with a constant $c > 0$ independent of $\ell \in \mathbb{N}$. But this is impossible since $t < 0$ and $d \ge 2$. The proof is complete. ■

In other situations of t, p, q we have following result.

Proposition 2.6. *Let $d \ge 2$.*
(i) *Let $1 \le p \le \infty$, $0 < q \le \infty$ and $t < 0$. Then we have*

$$B^t_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)\,.$$

(ii) *Let $t = 0$, $p \ne 2$, $1 < p < \infty$ and $\min(2,p) < q < \max(2,p)$. Then $B^0_{p,q}(\mathbb{R}^d)$ and $S^0_{p,q}B(\mathbb{R}^d)$ are not comparable.*
(iii) *Let $t = 0$, $p = 1$ and $1 < q < \infty$. Then $B^0_{1,q}(\mathbb{R}^d)$ and $S^0_{1,q}B(\mathbb{R}^d)$ are not comparable.*
(iv) *Let $t = 0$, $p = \infty$ and $1 < q < \infty$. Then $B^0_{\infty,q}(\mathbb{R}^d)$ and $S^0_{\infty,q}B(\mathbb{R}^d)$ are not comparable.*
(v) *Let $t < 0$, $0 < p < 1$ and $0 < q \le \infty$. Then $B^t_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}B(\mathbb{R}^d)$ are not comparable.*

Proof. Part (i) in case $1 \le q \le \infty$ follows by duality from Theorem 2.4, see Propositions 1.42 and 1.45. To cover also the case $0 < q < 1$ we can argue as in proof of Theorem 2.8 (sufficiency) replacing $B^{td}_{p,q}(\mathbb{R}^d)$ by $B^t_{p,q}(\mathbb{R}^d)$. Observe in this connection that

$$\sup_{j\in\mathbb{N}_0} \sum_{k\in\Delta_j} 2^{(|k|_1-j)tq} < \infty\,.$$

Parts (ii)-(iv) are immediate consequences of Theorems 2.4 and 2.8. Now we turn to the proof of (v). Theorem 2.4 yields $S^t_{p,q}B(\mathbb{R}^d) \not\hookrightarrow B^t_{p,q}(\mathbb{R}^d)$. It remains to prove $B^t_{p,q}(\mathbb{R}^d) \not\hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$. We will argue by contradiction. Therefore we assume $B^t_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$. This implies $\mathring{B}^t_{p,q}(\mathbb{R}^d) \hookrightarrow \mathring{S}^t_{p,q}B(\mathbb{R}^d)$. Propositions 1.42 and 1.45 yield

$$S^{-t+\frac{1}{p}-1}_{\infty,q'}B(\mathbb{R}^d) \hookrightarrow B^{-t+d(\frac{1}{p}-1)}_{\infty,q'}(\mathbb{R}^d)\,.$$

Since $d(\frac{1}{p}-1) > \frac{1}{p}-1$ it is enough to use $g_k(x) := e^{ikx}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, as test functions to disprove this embedding. ■

We summarize what is known about the relation of $B^t_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}B(\mathbb{R}^d)$ in the following figure.

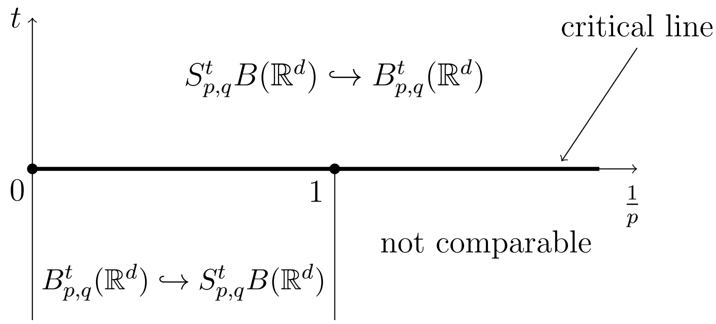

Figure 1. Comparison of $B^t_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}B(\mathbb{R}^d)$

2.2.2 The embedding of isotropic spaces into dominating mixed spaces

The somehow dual assertion of Lemma 2.5 reads as follows.

Lemma 2.7. *Let $d \geq 2$, $1 < p \leq \infty$ and $0 < q \leq \infty$. Then the embedding*

$$B^0_{p,q}(\mathbb{R}^d) \hookrightarrow S^0_{p,q}B(\mathbb{R}^d) \qquad \textit{implies} \qquad q \geq \max(p,2).$$

Proof. First the case $1 < p < \infty$ is carried out as Step 1 in the proof of Lemma 2.5. We consider the case $p = \infty$. Assume $B^0_{\infty,q}(\mathbb{R}^d) \hookrightarrow S^0_{\infty,q}B(\mathbb{R}^d)$ for some $q < \infty$. Then Propositions 1.49 and 1.51 yield with $\Theta = \frac{1}{2}$

$$S^0_{4,q_1}B(\mathbb{R}^d) = [S^0_{\infty,q}B(\mathbb{R}^d), S^0_{2,2}B(\mathbb{R}^d)]_\Theta \quad \text{and} \quad B^0_{4,q_1}(\mathbb{R}^d) = [B^0_{\infty,q}(\mathbb{R}^d), B^0_{2,2}(\mathbb{R}^d)]_\Theta.$$

Here $\frac{1}{q_1} = \frac{1}{2q} + \frac{1}{4}$. Prom Proposition 1.48 we have $B^0_{4,q_1}(\mathbb{R}^d) \hookrightarrow S^0_{4,q_1}B(\mathbb{R}^d)$ but this is a contradiction since $q_1 < 4$. The proof is complete. ■

Theorem 2.8. *Let $d \geq 2$, $0 < p,q \leq \infty$ and $t \in \mathbb{R}$. Then we have*

$$B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d) \tag{2.24}$$

if and only if one of the following conditions is satisfied

- $t > (\frac{1}{p}-1)_+$;

- $t=0$, $1<p\le\infty$ *and* $\max(2,p)\le q\le\infty$;
- $0<p\le 1$, $t=\frac{1}{p}-1$ *and* $q=\infty$.

Proof. We divide the proof into two parts.
Part I - sufficiency. *Step 1.* Let us prove (2.24) in case $t>\big(\frac{1}{p}-1\big)_+$. We put $u:=\min(1,p)$. From Part I (Step 1) of the proof of Theorem 2.4 we have

$$\varphi_k(x)=\sum_{j\in\square_k}\psi_j(x)\varphi_k(x)\,,\qquad x\in\mathbb{R}^d\,. \tag{2.25}$$

This identity yields

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\|^q\;=\;\sum_{k\in\mathbb{N}_0^d}2^{t|k|_1q}\Big\|\sum_{j\in\square_k}\mathcal{F}^{-1}\psi_j\varphi_k\mathcal{F}f\Big|L_p(\mathbb{R}^d)\Big\|^q.$$

Applying the inequality $|a+b|^u\le a^u+b^u$ and (2.2) we find

$$\begin{aligned}\|f|S^t_{p,q}B(\mathbb{R}^d)\|^q&\le\sum_{k\in\mathbb{N}_0^d}2^{t|k|_1q}\Big(\sum_{j\in\square_k}\big\|\mathcal{F}^{-1}\psi_j\varphi_k\mathcal{F}f|L_p(\mathbb{R}^d)\big\|^u\Big)^{q/u}\\&\le c_1\sum_{k\in\mathbb{N}_0^d}2^{t|k|_1q}\Big(\sum_{j\in\square_k}\big(2^{(jd-|k|_1)(\frac{1}{u}-1)}\|\mathcal{F}^{-1}\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|\big)^u\Big)^{q/u}.\end{aligned}$$

In view of (2.18) we obtain

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\|^q\;\le\;c_2\sum_{k\in\mathbb{N}_0^d}\sum_{j\in\square_k}2^{t|k|_1q}2^{(jd-|k|_1)(\frac{1}{u}-1)q}\|\mathcal{F}^{-1}\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|^q\,.$$

Consequently

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\|^q\;\le\;c_3\sum_{j=0}^{\infty}2^{jdtq}\|\mathcal{F}^{-1}\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|^q\sum_{k\in\Delta_j}2^{(jd-|k|_1)(\frac{1}{u}-1-t)q}\,.$$

It is easily derived from (2.3) and the restriction $t>\frac{1}{u}-1$ that

$$\sup_{j\in\mathbb{N}_0}\sum_{k\in\Delta_j}2^{(jd-|k|_1)(\frac{1}{u}-1-t)q}<\infty\,.$$

Hence we arrive at

$$\|f|S^t_{p,q}B(\mathbb{R}^d)\|^q\;\le\;c_1\sum_{j=0}^{\infty}2^{jdtq}\|\mathcal{F}^{-1}\psi_j\mathcal{F}f|L_p(\mathbb{R}^d)\|^q$$

which proves (2.24).
Step 2. Let $t=0$, $1<p\le\infty$ and $\max(2,p)\le q\le\infty$. We shall argue by duality. We have

$$S^0_{p',q'}B(\mathbb{R}^d)\hookrightarrow B^0_{p',q'}(\mathbb{R}^d)\,,$$

see Theorem 2.4. Now Propositions 1.42 and 1.45 can be used to prove the claim.
Step 3. Let $0 < p \le 1$, $t = \frac{1}{p} - 1$ and $q = \infty$. Applying (2.25) we find

$$\begin{aligned} \| f | S^{\frac{1}{p}-1}_{p,\infty} B(\mathbb{R}^d)\| &= \sup_{k \in \mathbb{N}_0^d} 2^{|k|_1(\frac{1}{p}-1)} \, \|\mathcal{F}^{-1}\varphi_k \mathcal{F} f | L_p(\mathbb{R}^d)\| \\ &= \sup_{k \in \mathbb{N}_0^d} 2^{|k|_1(\frac{1}{p}-1)} \Big\| \sum_{j \in \square_k} \mathcal{F}^{-1}\psi_j \varphi_k \mathcal{F} f \Big| L_p(\mathbb{R}^d)\Big\|. \end{aligned}$$

Making use of (2.2), this implies

$$\begin{aligned} \| f | S^{\frac{1}{p}-1}_{p,\infty} B(\mathbb{R}^d)\|^p &\le \sup_{k \in \mathbb{N}_0^d} 2^{|k|_1(\frac{1}{p}-1)p} \Big(\sum_{j \in \square_k} \big\|\mathcal{F}^{-1}\psi_j \varphi_k \mathcal{F} f \big| L_p(\mathbb{R}^d)\big\|^p \Big) \\ &\le c_1 \sup_{k \in \mathbb{N}_0^d} 2^{|k|_1(\frac{1}{p}-1)p} \Big(\sum_{j \in \square_k} 2^{(jd - |k|_1)(\frac{1}{p}-1)p} \, \|\mathcal{F}^{-1}\psi_j \mathcal{F} f | L_p(\mathbb{R}^d)\|^p \Big) \\ &= c_1 \sup_{k \in \mathbb{N}_0^d} \Big(\sum_{j \in \square_k} 2^{jd(\frac{1}{p}-1)p} \, \|\mathcal{F}^{-1}\psi_j \mathcal{F} f | L_p(\mathbb{R}^d)\|^p \Big). \end{aligned}$$

Taking into account (2.18), we obtain

$$\| f | S^{\frac{1}{p}-1}_{p,\infty} B(\mathbb{R}^d)\| \le c_2 \sup_{j \in \mathbb{N}_0} 2^{jd(\frac{1}{p}-1)} \, \|\mathcal{F}^{-1}\psi_j \mathcal{F} f | L_p(\mathbb{R}^d)\| = c_2 \| f | B^{d(\frac{1}{p}-1)}_{p,\infty}(\mathbb{R}^d)\| \, .$$

Part II - necessity. *Step 1.* Let $0 < p \le 1$ and $t = \frac{1}{p} - 1$. Assume that there is some $0 < q \le \infty$ such that $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$ holds. This leads to $\mathring{B}^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathring{S}^t_{p,q}B(\mathbb{R}^d)$. Now Propositions 1.42 and 1.45 yield

$$S^0_{\infty,q'}B(\mathbb{R}^d) \hookrightarrow B^0_{\infty,q'}(\mathbb{R}^d).$$

In view of Lemma 2.5 this implies $q' \le 1$, hence $q = \infty$.
Step 2. The necessity of the restrictions in case $t = 0$ follows from Lemma 2.7.
Step 3. It remains to deal with $t < (\frac{1}{p} - 1)_+$.
Step 3.1. Let $0 < p, q \le \infty$ and $t < 0$. We employ the test functions from Example 4. It is clear that the inequality

$$2^{\ell(t+1-\frac{1}{p})} \le c 2^{\ell(td+1-\frac{1}{p})} \iff 2^{t\ell} \le c 2^{dt\ell}$$

can not hold with a constant $c > 0$ independent of $\ell \in \mathbb{N}$ since $t < 0$ and $d \ge 2$. This implies $B^{td}_{p,q}(\mathbb{R}^d) \not\hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$, see (2.10) and (2.11).
Step 3.2. Let $0 < p < 1$ and $0 \le t < \frac{1}{p} - 1$. We assume $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$ which implies $\mathring{B}^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathring{S}^t_{p,q}B(\mathbb{R}^d)$. Propositions 1.42 and 1.45 yield

$$S^{-t+\frac{1}{p}-1}_{\infty,q'}B(\mathbb{R}^d) \hookrightarrow B^{d(-t+\frac{1}{p}-1)}_{\infty,q'}(\mathbb{R}^d) \, .$$

Since $d(-t+\frac{1}{p}-1) > -t+\frac{1}{p}-1$ it is enough to use $g_k(x) := e^{ikx}$, $k \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$, as test functions to disprove this embedding. ∎

Proposition 2.9. *Let $d \geq 2$.*
(i) *Let $0 < p, q \leq \infty$ and $t < 0$. Then we have*

$$S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow B^{td}_{p,q}(\mathbb{R}^d)\,.$$

(ii) *Let $0 < p < 1$, $0 < t < \frac{1}{p} - 1$ and $0 < q \leq \infty$. Then $B^{td}_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}B(\mathbb{R}^d)$ are not comparable.*
(iii) *Let $0 < p < 1$, $t = 0$ and $0 < q \leq p$. Then $S^0_{p,q}B(\mathbb{R}^d) \hookrightarrow B^0_{p,q}(\mathbb{R}^d)$ follows.*
(iv) *Let $0 < p < 1$, $t = 0$ and $p < q \leq \infty$. Then $B^0_{p,q}(\mathbb{R}^d)$ and $S^0_{p,q}B(\mathbb{R}^d)$ are not comparable.*

Proof. To prove (i) we follow the arguments used in proof of Theorem 2.4 (sufficiency) replacing $B^t_{p,q}(\mathbb{R}^d)$ by $B^{td}_{p,q}(\mathbb{R}^d)$.
Concerning (ii)-(iv), observe that $B^{td}_{p,q}(\mathbb{R}^d) \not\hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$ follows from Theorem 2.8. Now we split our investigations into two cases: $0 < t < \frac{1}{p} - 1$, $t = 0$.
Step 1. Let $0 < p < 1$ and $0 < t < \frac{1}{p} - 1$. $S^t_{p,q}B(\mathbb{R}^d) \not\hookrightarrow B^{td}_{p,q}(\mathbb{R}^d)$ follows from Example 4, see (2.10) and (2.11).
Step 2. Let $0 < p < 1$ and $t = 0$. Theorem 2.4 yields

$$S^0_{p,q}B(\mathbb{R}^d) \hookrightarrow B^0_{p,q}(\mathbb{R}^d) \qquad \Longleftrightarrow \qquad 0 < q \leq p\,.$$

Hence, in case $q > p$ the spaces $S^0_{p,q}B(\mathbb{R}^d)$ and $B^0_{p,q}(\mathbb{R}^d)$ are not comparable. ■

We summarize the results in this section in the following figure.

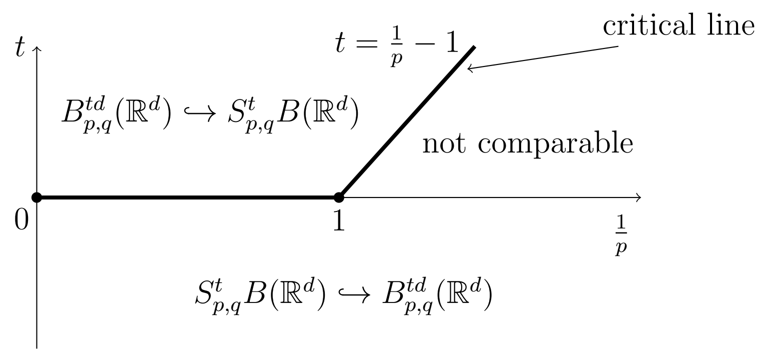

Figure 2. Comparison of $S^t_{p,q}B(\mathbb{R}^d)$ and $B^{td}_{p,q}(\mathbb{R}^d)$

2.3 The case of Triebel-Lizorkin spaces

2.3.1 The embedding of dominating mixed spaces into isotropic spaces

Theorem 2.10. *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$ and $t \geq 0$. Then we have*

$$S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow F^t_{p,q}(\mathbb{R}^d)$$

if one of the following conditions is satisfied:

- $t > 0$;
- $t = 0$, $1 < p$ *and* $q \leq 2$;

- $t = 0$, $p \le 1$ *and* $q < 2$.

Remark 2.11. We recall that $S^0_{p,2}F(\mathbb{R}^d) = F^0_{p,2}(\mathbb{R}^d) = L_p(\mathbb{R}^d)$, $1 < p < \infty$, in the sense of equivalent norms. This is a consequence of certain Littlewood-Paley assertions, see Nikol'skij [78, Section 1.5.6]. This identity does not extend to $p \le 1$. Here we conjecture

$$S^0_{p,2}F(\mathbb{R}^d) \hookrightarrow F^0_{p,2}(\mathbb{R}^d), \qquad 0 < p \le 1.$$

Proof. We use the notations in Part I (Step 1) of the proof of Theorem 2.4.
Step 1. We prove the case $t > 0$. From (2.19) we have

$$\|f|F^t_{p,q}(\mathbb{R}^d)\| = \bigg\|\bigg(\sum_{j=0}^{\infty}\Big|\sum_{k\in\Delta_j} 2^{tj-t|k|_1}2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\Big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|. \tag{2.26}$$

If $q \le 1$, then

$$\begin{aligned} \|f|F^t_{p,q}(\mathbb{R}^d)\| &\le \bigg\|\bigg(\sum_{j=0}^{\infty}\sum_{k\in\Delta_j}\big|2^{t(j-|k|_1)}2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\| \\ &\le c_1\bigg\|\bigg(\sum_{j=0}^{\infty}\sum_{k\in\Delta_j}\big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|. \end{aligned} \tag{2.27}$$

The last inequality is due to $2^{t(j-|k|_1)} \le C$ for all $j \in \mathbb{N}_0$ and $k \in \Delta_j$. Using Hölder's inequality for the case $q > 1$ we obtain

$$\Big|\sum_{k\in\Delta_j} 2^{tj-t|k|_1}2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\Big| \le \bigg(\sum_{k\in\Delta_j}\big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg(\sum_{k\in\Delta_j}2^{t(j-|k|_1)q'}\bigg)^{1/q'},$$

here $\frac{1}{q} + \frac{1}{q'} = 1$. Because of $t > 0$ and (2.3) the second sum on the right-hand side is uniformly bounded, see also (2.22). Consequently, we obtain from (2.26)

$$\|f|F^t_{p,q}(\mathbb{R}^d)\| \le c_2\bigg\|\bigg(\sum_{j=0}^{\infty}\sum_{k\in\Delta_j}\big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|.$$

From this and (2.27) we have proved that

$$\begin{aligned} \|f|F^t_{p,q}(\mathbb{R}^d)\| &\le c_3\bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\sum_{j\in\square_k}\big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\| \\ &\le c_4\sum_{i=-1}^{1}\bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_{j+i}\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|, \end{aligned}$$

where $j = |k|_\infty$, see (2.3). We estimate the term with $i = 0$. The terms with $i = \pm 1$ can be treated in a similar way. Let $\{\tilde{\varphi}_k\}_{k\in\mathbb{N}_0^d}$ be the system defined in the proof of Lemma 1.44. Then we have

$$\begin{aligned} &\bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\| \\ &\qquad = \bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\big|\mathcal{F}^{-1}\tilde{\varphi}_k\psi_j\mathcal{F}\big[2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}f\big]\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|. \end{aligned}$$

Applying Lemma 1.11 with $M_k = \tilde{\varphi}_k \psi_j$ we obtain

$$
\begin{aligned}
&\Big\|\Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| \\
&\quad \le C_1 \sup_{k\in\mathbb{N}_0^d} \|(\tilde{\varphi}_k\psi_j)(2^{k+\bar{1}}\diamond\cdot)|S_2^rW(\mathbb{R}^d)\| \Big\|\Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|
\end{aligned} \tag{2.28}
$$

for $r \in \mathbb{N}$, $r > \frac{1}{\min(p,q)} + \frac{1}{2}$. To estimate the factor $\|\ldots|S_2^rW(\mathbb{R}^d)\|$ we consider the case $k \in \mathbb{N}_0^d$ with $\min_{i=1,\ldots,d} k_i \ge 1$. We have

$$
\begin{aligned}
\|(\tilde{\varphi}_k\psi_j)(2^{k+\bar{1}}\diamond\cdot)|S_2^rW(\mathbb{R}^d)\| &\asymp \|\tilde{\varphi}_{\bar{1}}(4\cdot)\psi_1(2^{k-\bar{j}+\bar{2}}\diamond\cdot)|S_2^rW(\mathbb{R}^d)\| \\
&\le C_2\|\psi_1(2^{k-\bar{j}+\bar{2}}\diamond\cdot)|C^r_{\mathrm{mix}}(\mathbb{R}^d)\|\cdot\|\tilde{\varphi}_{\bar{1}}(4\cdot)|S_2^rW(\mathbb{R}^d)\|.
\end{aligned}
$$

Since $k_i - j = k_i - |k|_\infty \le 0$, $i = 1,\ldots,d$, for every multi-index $\alpha \in \mathbb{N}_0^d$ with $\alpha_i \le r$ we have

$$
\sup_{x\in\mathbb{R}^d} |D^\alpha(\psi_1(2^{k-\bar{j}+\bar{2}}\diamond x))| = \sup_{x\in\mathbb{R}^d} 2^{\alpha(k-\bar{j}+\bar{2})}|(D^\alpha\psi_1)(2^{k-\bar{j}+\bar{2}}\diamond x)| \le C_\alpha
$$

which implies

$$
\sup_{k\in\mathbb{N}^d} \|(\tilde{\varphi}_k\psi_j)(2^{k+\bar{1}}\diamond\cdot)|S_2^rW(\mathbb{R}^d)\| \le C_3\,.
$$

By a modification of the above argument for $k \in \mathbb{N}_0^d$ with $\min_{i=1,\ldots,d} k_i = 0$ we conclude that the first term on the right-hand side of (2.28) is uniformly bounded. Consequently,

$$
\Big\|\Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{t|k|_1}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| \le C_4\|f|S^t_{p,q}F(\mathbb{R}^d)\|.
$$

This implies $S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow F^t_{p,q}(\mathbb{R}^d)$.
Step 2. The case $t = 0$. If $0 < q \le 1$ we obtain from (2.27) with $t = 0$

$$
\|f|F^t_{p,q}(\mathbb{R}^d)\| \;\;\le\;\; \Big\|\Big(\sum_{j=0}^\infty\sum_{k\in\Delta_j} \big|\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|.
$$

Next step is carried out as Step 1. We now consider the case $1 < q < 2$ by employing the interpolation property. For $0 < p < \infty$ and $1 < q < 2$ there exist $\Theta \in (0,1)$, $0 < p_0 < \infty$ and $1 < p_1 < \infty$ such that

$$
\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \qquad \text{and} \qquad \frac{1}{q} = \frac{1-\Theta}{1} + \frac{\Theta}{2}.
$$

Propositions 1.49 and 1.51 yield

$$
S^0_{p,q}F(\mathbb{R}^d) = [S^0_{p_0,1}F(\mathbb{R}^d), S^0_{p_1,2}F(\mathbb{R}^d)]_\Theta \quad \text{and} \quad F^0_{p,q}(\mathbb{R}^d) = [F^0_{p_0,1}(\mathbb{R}^d), F^0_{p_1,2}(\mathbb{R}^d)]_\Theta\,.
$$

Finally, the claim follows from Proposition 1.48. The proof is complete. ■

Lemma 2.12. *Let $d \geq 2$, $0 < p < \infty$ and $0 < q \leq \infty$. Then the embedding*

$$S^0_{p,q}F(\mathbb{R}^d) \hookrightarrow F^0_{p,q}(\mathbb{R}^d) \qquad \textit{implies} \qquad q \leq 2.$$

Proof. *Step 1.* The case $1 < p < \infty$. Employing the test functions in Example 1, a similar argument as in Substep 1.1 of the proof of Lemma 2.5 yields $q \leq 2$.
Step 2. The case $0 < p \leq 1$. Assume that $S^0_{p,q}F(\mathbb{R}^d) \hookrightarrow F^0_{p,q}(\mathbb{R}^d)$ with $0 < p \leq 1$ and $2 < q \leq \infty$. Then we can find a triple (p_1, q_1, Θ) such that $\Theta \in (0,1)$, $1 < p_1 < 2 < q_1 < \infty$,

$$\frac{1}{p_1} = \frac{\Theta}{p} + \frac{1-\Theta}{2} \qquad \text{and} \qquad \frac{1}{q_1} = \frac{\Theta}{q} + \frac{1-\Theta}{2}.$$

In a view of Propositions 1.48, 1.49 and 1.51 we conclude $S^0_{p_1,q_1}F(\mathbb{R}^d) \hookrightarrow F^0_{p_1,q_1}(\mathbb{R}^d)$. This is a contradiction with Step 1 since $p_1 > 1$ and $q_1 > 2$. ■

Theorem 2.13. *Let $d \geq 2$, $1 < p < \infty$, $1 \leq q \leq \infty$ and $t \in \mathbb{R}$. Then*

$$S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow F^t_{p,q}(\mathbb{R}^d)$$

if and only if either $t > 0$ or $t = 0$ and $q \leq 2$.

Proof. As a consequence of Theorem 2.10 and Lemma 2.12 it will be enough to consider the case $t < 0$. We assume $S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow F^t_{p,q}(\mathbb{R}^d)$ if $t < 0$. But now the test functions from Example 5 can be used to disprove this embedding. ■

In the case $t < 0$ we have the following.

Proposition 2.14. *Let $d \geq 2$ and $t < 0$.*
(i) *If $1 < p < \infty$ and $1 \leq q \leq \infty$, then $F^t_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$.*
(ii) *If $0 < p < 1$, $0 < q \leq \infty$, then $F^t_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}F(\mathbb{R}^d)$ are not comparable.*

Proof. *Step 1.* Proof of (i). Theorem 2.10 implies $\mathring{S}^t_{p,q}F(\mathbb{R}^d) \hookrightarrow \mathring{F}^t_{p,q}(\mathbb{R}^d)$ if $t > 0$. Propositions 1.42 and 1.47 yield $F^{-t}_{p',q'}(\mathbb{R}^d) \hookrightarrow S^{-t}_{p',q'}F(\mathbb{R}^d)$, if $1 < p < \infty$ and $1 \leq q \leq \infty$.
Step 2. Proof of (ii). Since $-t + \frac{1}{p} - 1 < -t + d(\frac{1}{p} - 1)$ and $t < 0$ we can use $g_k = e^{ikx}$ as test functions to prove that $S^{-t+\frac{1}{p}-1}_{\infty,\infty}B(\mathbb{R}^d)$ and $B^{-t+d(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^d)$ are not comparable. Then, from Propositions 1.42 and 1.47, we can conclude that $\mathring{S}^t_{p,q}F(\mathbb{R}^d)$ and $\mathring{F}^t_{p,q}(\mathbb{R}^d)$ are incomparable and therefore $S^t_{p,q}F(\mathbb{R}^d)$ and $F^t_{p,q}(\mathbb{R}^d)$ as well. This finishes the proof. ■

As in the case of Besov spaces, we summarize the relation between $F^t_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}F(\mathbb{R}^d)$ $(1 \leq q \leq \infty)$ in the following figure.

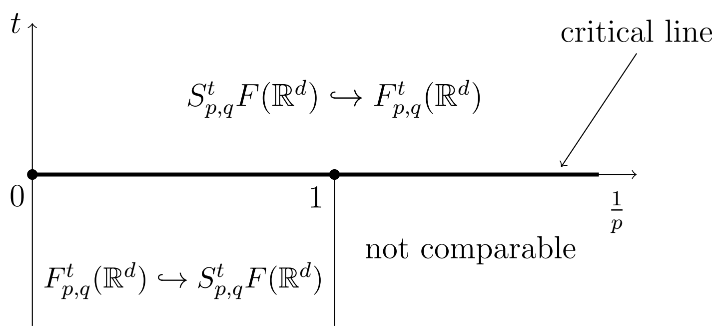

Figure 3. Comparison of $S^t_{p,q}F(\mathbb{R}^d)$ and $F^t_{p,q}(\mathbb{R}^d)$

2.3.2 The embedding of isotropic spaces into dominating mixed spaces

Theorem 2.15. *Let $d \geq 2$, $0 < p < \infty$, $0 < q \leq \infty$ and $t \geq 0$. Then we have*

$$F^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$$

if one of the following conditions is satisfied:

- $t > \big(\frac{1}{\min(p,q)} - 1\big)_+$ *and* $0 < q < \infty$;
- $t = 0$, $p > 1$ *and* $2 \leq q \leq \infty$.

Proof. The claim for the case $t = 0$ is a consequence of Theorem 2.10 and duality, see Propositions 1.42 and 1.47. The proof in case $t > \big(\frac{1}{\min(p,q)} - 1\big)_+$ will be divided into several steps.
Step 1. We prove that the embedding holds true if $0 < p < \infty$, $0 < q \leq \infty$ and $t > \frac{1}{\min(p,q)}$. Let $\tau = \min(1,p,q)$. From (2.25) and $\Box_k \asymp 1$ we obtain

$$\begin{aligned} \|f|S^t_{p,q}F(\mathbb{R}^d)\|^\tau &= \bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\Big|\sum_{j\in\Box_k} 2^{|k|_1 t}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\Big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|^\tau \\ &\leq \sum_{i=-1}^{1}\bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\big|2^{|k|_1 t}\mathcal{F}^{-1}\varphi_k\psi_{j+i}\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|^\tau, \end{aligned} \tag{2.29}$$

where again $j = |k|_\infty$. It will be enough to deal with the term for $i = 0$. The other terms can be treated similarly. Since $t > \frac{1}{\min(p,q)}$ we can write $t = a + \varepsilon$ with $a > \frac{1}{\min(p,q)}$ and $\varepsilon > 0$. By denoting

$$g_k = \mathcal{F}^{-1}\big[2^{(|k|_1 - jd)\varepsilon}2^{jtd}\psi_j\mathcal{F}f\big], \qquad k \in \mathbb{N}_0^d, \quad |k|_\infty = j$$

we can rewrite as follows

$$\begin{aligned} &\bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\big|2^{|k|_1 t}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\| \\ &\qquad = \bigg\|\bigg(\sum_{k\in\mathbb{N}_0^d}\big|2^{(|k|_1 - jd)a}\mathcal{F}^{-1}\varphi_k\mathcal{F}g_k\big|^q\bigg)^{1/q}\bigg|L_p(\mathbb{R}^d)\bigg\|. \end{aligned} \tag{2.30}$$

We consider the maximal function $P_{2^{\bar{j}+\bar{1}},a}g_k(x)$. Then we obtain

$$\begin{aligned} \big|\big(\mathcal{F}^{-1}\varphi_k\mathcal{F}g_k\big)(x-z)\big| &\leq (2\pi)^{-d/2}\int_{\mathbb{R}^d}\big|(\mathcal{F}^{-1}\varphi_k)(x-z-y)\big|\cdot|g_k(y)|\,\mathrm{d}y \\ &\leq (2\pi)^{-d/2}P_{2^{\bar{j}+\bar{1}},a}g_k(x)\int_{\mathbb{R}^d}\big|(\mathcal{F}^{-1}\varphi_k)(x-z-y)\big|\prod_{i=1}^{d}(1+|2^{j+1}(x_i-y_i)|^a)\,\mathrm{d}y. \end{aligned}$$

The elementary inequality

$$(1+|2^{j+1}(x_i-y_i)|^a) \leq 2^a(1+|2^{j+1}z_i|^a)(1+|2^{j+1}(x_i-z_i-y_i)|^a), \quad i = 1,\dots,d,$$

and change of variable lead to

$$\frac{\big|\big(\mathcal{F}^{-1}\varphi_k\mathcal{F}g_k\big)(x-z)\big|}{\prod_{i=1}^d(1+|2^{j+1}z_i|^a)} \le C_1\, P_{2^{\bar{j}+\bar{1}},a}g_k(x)\int\limits_{\mathbb{R}^d}\big|(\mathcal{F}^{-1}\varphi_k)(y)\big|\prod_{i=1}^d(1+|2^{j+1}y_i|^a)\,\mathrm{d}y$$

We temporarily assume $k\ge\bar{1}$. Then

$$\int\limits_{\mathbb{R}^d}\big|(\mathcal{F}^{-1}\varphi_k)(y)\big|\prod_{i=1}^d(1+|2^{j+1}y_i|^a)\,\mathrm{d}y=\prod_{i=1}^d\int\limits_{\mathbb{R}}\big|\mathcal{F}^{-1}\varphi_1(\xi)\big|(1+2^{j+2-k_i}|\xi|)^a\,\mathrm{d}\xi$$

follows. Since $k_i\le|k|_\infty=j$ and $\mathcal{F}^{-1}\varphi_1\in\mathcal{S}(\mathbb{R})$ we have

$$\begin{aligned}\int\limits_{\mathbb{R}}\big|\mathcal{F}^{-1}\varphi_1(\xi)\big|(1+2^{j+2-k_i}|\xi|)^a\,\mathrm{d}\xi &= 2^{(j-k_i)a}\int\limits_{\mathbb{R}}\big|\mathcal{F}^{-1}\varphi_1(\xi)\big|(2^{k_i-j}+4|\xi|)^a\,\mathrm{d}\xi\\ &\le C_2 2^{(j-k_i)a}.\end{aligned}$$

By obvious modifications this estimate also holds for $k\in\mathbb{N}_0^d$ with $\min_{i=1,\ldots,d}k_i=0$. Consequently

$$2^{(|k|_1-jd)a}\frac{\big|\big(\mathcal{F}^{-1}\varphi_k\mathcal{F}g_k\big)(x-z)\big|}{\prod_{i=1}^d(1+2^{j+1}|z_i|)^a}\le C_3\;P_{2^{\bar{j}+\bar{1}},a}g_k(x)$$

with a constant C_3 independent of x and $\{g_k\}_{k\in\mathbb{N}_0^d}$. Obviously, we have

$$\begin{aligned}2^{(|k|_1-jd)a}\big|\big(\mathcal{F}^{-1}\varphi_k\mathcal{F}g_k\big)(x)\big| &\le \sup_{z\in\mathbb{R}^d}\frac{2^{(|k|_1-jd)a}\big|\big(\mathcal{F}^{-1}\varphi_k\mathcal{F}g_k\big)(x-z)\big|}{\prod_{i=1}^d(1+|2^{j+1}z_i|^a)}\\ &\le C_3P_{2^{\bar{j}+\bar{1}},a}g_k(x),\end{aligned}$$

which results in the estimate

$$\Big\|\Big(\sum_{k\in\mathbb{N}_0^d}\big|2^{|k|_1t}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|\le C_3\Big\|\Big(\sum_{k\in\mathbb{N}_0^d}\big|P_{2^{\bar{j}+\bar{1}},a}g_k(\cdot)\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|,$$

see (2.30). Now, applying Theorem 1.10 for $\{g_k\}_{k\in\mathbb{N}_0^d}$ with $b_{k_i}=2^{j+1}$, $i=1,\ldots,d$, we obtain

$$\begin{aligned}&\Big\|\Big(\sum_{k\in\mathbb{N}_0^d}\big|2^{|k|_1t}\mathcal{F}^{-1}\varphi_k\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|\\ &\qquad\le C_4\Big\|\Big(\sum_{k\in\mathbb{N}_0^d}\big|\mathcal{F}^{-1}2^{(|k|_1-jd)\varepsilon}2^{jtd}\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|\\ &\qquad\le C_4\Big\|\Big(\sum_{k\in\mathbb{N}_0^d}\sum_{j\in\square_k}\big|\mathcal{F}^{-1}2^{(|k|_1-jd)\varepsilon}2^{jtd}\psi_j\mathcal{F}f\big|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|\\ &\qquad= C_4\Big\|\Big(\sum_{j=0}^\infty\big|2^{jtd}\mathcal{F}^{-1}\psi_j\mathcal{F}f\big|^q\sum_{k\in\Delta_j}2^{(|k|_1-jd)\varepsilon q}\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|.\end{aligned}$$

The condition $\varepsilon > 0$ results in

$$\Big\| \Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{|k|_1 t} \mathcal{F}^{-1}\varphi_k \psi_j \mathcal{F} f\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \le C_5 \Big\| \Big(\sum_{j=0}^{\infty} \big|2^{jtd} \mathcal{F}^{-1} \psi_j \mathcal{F} f\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\|$$
$$= C_5 \|f|F^{td}_{p,q}(\mathbb{R}^d)\|,$$

see (2.3). Inserting this into (2.29) and carrying out the other terms in the same way, the claim follows.

Substep 2. We prove that the embedding holds true if $1 < p < \infty$, $1 < q \le \infty$ and $t > 0$. This time we use Proposition 1.4. We begin from (2.29). As above it will be enough to deal with $j = |k|_\infty$. Applying Proposition 1.4 in connection with decomposition (1.13) we have found

$$\begin{aligned}
&\Big\| \Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{|k|_1 t} \mathcal{F}^{-1}\varphi_k \psi_j \mathcal{F} f\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \\
&\qquad = \Big\| \Big(\sum_{k\in\mathbb{N}_0^d} \big|\mathcal{F}^{-1}\varphi_k \mathcal{F}\big[2^{|k|_1 t} \mathcal{F}^{-1}\psi_j \mathcal{F} f\big]\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \\
&\qquad \le C_6 \Big\| \Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{|k|_1 t} \mathcal{F}^{-1}\psi_j \mathcal{F} f\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \\
&\qquad \le C_6 \Big\| \Big(\sum_{j=0}^{\infty} \big|2^{jtd} \mathcal{F}^{-1}\psi_j \mathcal{F} f\big|^q \sum_{k\in\Delta_j} 2^{(|k|_1 - jd)tq} \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\|.
\end{aligned}$$

Because of $t > 0$ we conclude that

$$\Big\| \Big(\sum_{k\in\mathbb{N}_0^d} \big|2^{|k|_1 t} \mathcal{F}^{-1}\varphi_k \psi_j \mathcal{F} f\big|^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \le c_3 \|f|F^{td}_{p,q}(\mathbb{R}^d)\|.$$

In view of (2.29) the claim follows.

Step 3. We interpolate the results in Steps 1 and 2 to prove for $0 < p, q < \infty$ and $t > \big(\frac{1}{\min(p,q)} - 1\big)_+$. Assume that $\min(p,q) \le 1$ and $p \le q$. Since $t > \frac{1}{p} - 1$ we choose $p_0 > 1$, $0 < \Theta < 1$ and $\varepsilon > 0$ such that

$$t = \varepsilon + \frac{1}{p} - \frac{1}{p_0} + \frac{\Theta}{p_0}.$$

Next we define (p_0, q_0), (p_1, q_1) by

$$\frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{p_1} \quad \text{and} \quad \frac{p}{q} = \frac{p_0}{q_0} = \frac{p_1}{q_1}.$$

Now we put $t_0 = \varepsilon$ and $t_1 = \frac{1}{\min(p_1,q_1)} + \varepsilon = \frac{1}{p_1} + \varepsilon$ since $p_1 \le q_1$. Hence we obtain $t = (1-\Theta)t_0 + \Theta t_1$. Propositions 1.49 and 1.51 yield

$$F^{td}_{p,q}(\mathbb{R}^d) = [F^{t_0 d}_{p_0,q_0}(\mathbb{R}^d), F^{t_1 d}_{p_1,q_1}(\mathbb{R}^d)]_\Theta \quad \text{and} \quad S^t_{p,q}F(\mathbb{R}^d) = [S^{t_0}_{p_0,q_0}F(\mathbb{R}^d), S^{t_1}_{p_1,q_1}F(\mathbb{R}^d)]_\Theta\,.$$

From Proposition 1.48, Steps 1 and 2 we find $F^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$. The case $\min(p,q) \le 1$ and $q \le p$ can be argued similarly by interchanging the roles of p and q. We finish the proof. ∎

Remark 2.16. In view of the proof of Theorem 2.15 we know that the embedding $F^{td}_{p,\infty}(\mathbb{R}^d) \hookrightarrow S^t_{p,\infty}F(\mathbb{R}^d)$ holds if either $1 < p < \infty$ and $t > 0$ or $0 < p \le 1$ and $t > \frac{1}{p}$. Note, that the interpolation argument in Step 3 does not extend to the case $q_0 = q_1 = \infty$ since it is known that

$$[F^{t_0 d}_{p_0,\infty}(\mathbb{R}^d), F^{t_1 d}_{p_1,\infty}(\mathbb{R}^d)]_\Theta \neq F^{td}_{p,\infty}(\mathbb{R}^d)$$

if $F^{t_0 d}_{p_0,\infty}(\mathbb{R}^d) \neq F^{t_1 d}_{p_1,\infty}(\mathbb{R}^d)$, see [144] and Remark 1.50. To overcome this situation, one could apply the $\pm$ method of Gustavsson and Peetre, denoted by $\langle\,\cdot\,,\,\cdot\,,\Theta\rangle$, to obtain

$$\langle F^{t_0 d}_{p_0,\infty}(\mathbb{R}^d), F^{t_1 d}_{p_1,\infty}(\mathbb{R}^d), \Theta\rangle = F^{td}_{p,\infty}(\mathbb{R}^d),$$

see again [144]. However, the assertion

$$\langle S^{t_0}_{p_0,\infty}F(\mathbb{R}^d), S^{t_1}_{p_1,\infty}F(\mathbb{R}^d), \Theta\rangle = S^t_{p,\infty}F(\mathbb{R}^d)$$

has not been proved in the literature.

By using the similar argument as in the proof of Theorem 2.13 we can conclude the following.

Theorem 2.17. *Let $d \ge 2$, $1 < p < \infty$, $1 \le q \le \infty$ and $t \in \mathbb{R}$. Then*

$$F^{td}_{p,q} \hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$$

if and only if either $t > 0$ or $t = 0$ and $q \ge 2$.

Proof. By Theorem 2.15 and Lemma 2.12 it will be enough to deal with $t < 0$. We assume that $F^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$ if $t < 0$. Applying Example 6 with $a_j := \delta_{j,\ell}$ we come to a contradiction. ∎

In addition we have the result.

Proposition 2.18. *Let $d \ge 2$.*
(i) *Let $0 < p < 1$, $0 < q \le \infty$ and $0 < t \le \frac{1}{p} - 1$. Then $S^t_{p,q}F(\mathbb{R}^d)$ and $F^{td}_{p,q}(\mathbb{R}^d)$ are not comparable.*
(ii) *Let $0 < p < \infty$, $0 < q \le \infty$ and $t < 0$. Then $S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow F^{td}_{p,q}(\mathbb{R}^d)$ follows.*

Proof. *Step 1.* Proof of (i). Assuming $F^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$ we get $\mathring{F}^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathring{S}^t_{p,q}F(\mathbb{R}^d)$ and therefore

$$S^{-t+\frac{1}{p}-1}_{\infty,\infty}B(\mathbb{R}^d) \hookrightarrow B^{-td+d(\frac{1}{p}-1)}_{\infty,\infty}(\mathbb{R}^d),$$

see Propositions 1.42 and 1.47. Since $-td + d(\frac{1}{p} - 1) > -t + \frac{1}{p} - 1 \ge 0$ this embedding is impossible (again it will be enough to use e^{ikx} as test functions). Hence $F^{td}_{p,q}(\mathbb{R}^d) \not\hookrightarrow S^t_{p,q}F(\mathbb{R}^d)$. By employing the test function in Example 6 with $a_j := \delta_{j,\ell}$ we can disprove the embedding $S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow F^{td}_{p,q}(\mathbb{R}^d)$.
Step 2. Proof of (ii). We argue as in the proof of Theorem 2.10 replacing $F^t_{p,q}(\mathbb{R}^d)$ by $F^{td}_{p,q}(\mathbb{R}^d)$ and taking into account that $t < 0$. The proof is complete. ∎

We summarize the relation between $F^{td}_{p,q}(\mathbb{R}^d)$ and $S^t_{p,q}F(\mathbb{R}^d)$ $(1 \le q \le \infty)$ in the following figure.

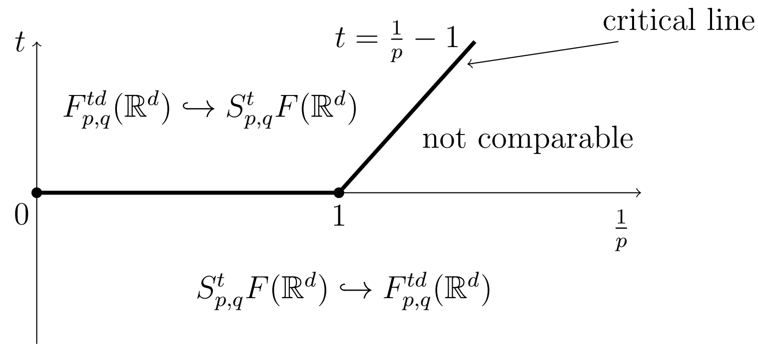

Figure 4. Comparison of $S^t_{p,q}F(\mathbb{R}^d)$ and $F^{td}_{p,q}(\mathbb{R}^d)$

2.4 The optimality

The embeddings in Theorems 2.4, 2.8, 2.10 and 2.15 are optimal in the following sense.

Theorem 2.19. *Let $0 < p_0, p, q_0, q \le \infty$ (with $p, p_0 < \infty$ in the F-case) and $t_0, t \in \mathbb{R}$. Let p, q and t be fixed.*
(i) *Within all spaces $S^{t_0}_{p_0,q_0}A(\mathbb{R}^d)$ satisfying $S^{t_0}_{p_0,q_0}A(\mathbb{R}^d) \hookrightarrow A^t_{p,q}(\mathbb{R}^d)$ the class $S^t_{p,q}A(\mathbb{R}^d)$ is the largest.*
(ii) *Within all spaces $A^{t_0}_{p_0,q_0}(\mathbb{R}^d)$ satisfying $A^{t_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}A(\mathbb{R}^d)$ the class $A^{td}_{p,q}A(\mathbb{R}^d)$ is the largest.*
(iii) *Within all spaces $S^{t_0}_{p_0,q_0}A(\mathbb{R}^d)$ satisfying $A^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0}A(\mathbb{R}^d)$ the class $S^t_{p,q}A(\mathbb{R}^d)$ is the smallest.*

Before proving, let us recall some well-known results about embeddings of Besov and Lizorkin - Triebel spaces.

Lemma 2.20. *Let $t, t_0 \in \mathbb{R}$, $0 < q, q_0 \le \infty$ and $0 < p \le p_0 \le \infty$.*
(i) *The embedding $B^t_{p,q}(\mathbb{R}^d) \hookrightarrow B^{t_0}_{p_0,q_0}(\mathbb{R}^d)$ holds if and only if either*

$$t_0 - \frac{d}{p_0} < t - \frac{d}{p} \quad \text{or} \quad t_0 - \frac{d}{p_0} = t - \frac{d}{p} \quad \text{and} \quad q \le q_0 \,.$$

(ii) *The embedding $S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0}B(\mathbb{R}^d)$ holds if and only if either*

$$t_0 - \frac{1}{p_0} < t - \frac{1}{p} \quad \text{or} \quad t_0 - \frac{1}{p_0} = t - \frac{1}{p} \quad \text{and} \quad q \le q_0 \,.$$

Remark 2.21. The results in Lemma 2.20 for isotropic spaces have a certain history. For the first time they have been proved by Taibleson in his series of papers [112]-[114], but see also [130, Theorem 2.7.1] and [105]. In case of the Besov spaces of dominating mixed smoothness we refer to [104, Section 2.4.1] and [103, 47].

Lemma 2.22. *Let $t, t_0 \in \mathbb{R}$, $0 < p < p_0 < \infty$ and $0 < q, q_0 \le \infty$.*
(i) *The embedding $F^t_{p,q}(\mathbb{R}^d) \hookrightarrow F^{t_0}_{p_0,q_0}(\mathbb{R}^d)$ holds if and only if $t_0 - \frac{d}{p_0} \le t - \frac{d}{p}$.*
(ii) *The embedding $S^t_{p,q}F(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0}F(\mathbb{R}^d)$ holds if and only if $t_0 - \frac{1}{p_0} \le t - \frac{1}{p}$.*

Remark 2.23. Note, that in case $p = p_0$ and $t = t_0$, the embedding $F^t_{p,q}(\mathbb{R}^d) \hookrightarrow F^t_{p,q_0}(\mathbb{R}^d)$, holds true if and only if $q \le q_0$. A similar statement is true for Lizorkin-Triebel spaces of dominating mixed smoothness. The assertion (i) in Lemma 2.22 can be found in [51], [130, Theorem 2.7.1] (sufficiency) and in [105] (necessity). In case of Triebel-Lizorkin spaces of dominating mixed smoothness we refer to [104, Section 2.4.1] and [103, 47] (sufficiency). Necessity can be obtained by employing the test functions in Example 4.

Proof of Theorem 2.19. *Step 1.* We prove for the case of Besov spaces.
Substep 1.1. Proof of (i). Assuming $S^{t_0}_{p_0,q_0}B(\mathbb{R}^d) \hookrightarrow B^t_{p,q}(\mathbb{R}^d)$ Lemma 2.3 shows that this implies $p_0 \le p$. Next we apply Example 4 to derive some relations between p_0 and p. We have

$$\| f_\ell \,|B^t_{p,q}(\mathbb{R}^d)\| = C\, 2^{\ell(t+1-\frac{1}{p})} \qquad \text{and} \qquad \| f_\ell \,|S^{t_0}_{p_0,q_0}B(\mathbb{R}^d)\| = C 2^{\ell(t_0+1-\frac{1}{p_0})}$$

see (2.10) and (2.11). The assumed embedding implies

$$t+1-\frac{1}{p} \le t_0+1-\frac{1}{p_0} \qquad \text{or} \qquad t-\frac{1}{p} \le t_0-\frac{1}{p_0}.$$

In case $t-\frac{1}{p} = t_0 - \frac{1}{p_0}$ we employ Example 4 again with $a_j := 2^{-j(t+1-1/p)}$, $j = 1, \ldots, \ell$, see (2.12). As a consequence of the embedding we derive $q_0 \le q$. This implies

$$S^{t_0}_{p_0,q_0}B(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)\,,$$

see Lemma 2.20.
Substep 1.2. Proof of (ii). Assuming $B^{t_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$ Lemma 2.3 implies $p_0 \le p$. Next we employ Example 5 to obtain

$$\| f_\ell \,|S^t_{p,q}B(\mathbb{R}^d)\| = C\, 2^{\ell d(t+1-\frac{1}{p})} \qquad \text{and} \qquad \| f_\ell \,|B^{t_0}_{p_0,q_0}(\mathbb{R}^d)\| = C\, 2^{\ell d(\frac{t_0}{d}+1-\frac{1}{p_0})}$$

with $C > 0$ independent of ℓ, see (2.13) and (2.14). The embedding $B^{t_0}_{p_0,q_0}(\mathbb{R}^d) \hookrightarrow S^t_{p,q}B(\mathbb{R}^d)$ yields

$$\frac{t_0}{d} - \frac{1}{p_0} \ge t - \frac{1}{p} \qquad \text{or} \qquad t_0 - \frac{d}{p_0} \ge dt - \frac{d}{p}\,.$$

Now, if $t_0 - \frac{d}{p_0} = dt - \frac{d}{p}$ we use test function h_ℓ from Example 5 with $a_j := 2^{-jd(t_0/d+1-1/p_0)}$ to obtain $q_0 \le q$, see (2.15) and (2.16). All together we conclude

$$S^{t_0}_{p_0,q_0}B(\mathbb{R}^d) \hookrightarrow S^{td}_{p,q}B(\mathbb{R}^d)\,.$$

see Lemma 2.20.
Substep 1.3. Proof of (iii). Assuming $B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0}B(\mathbb{R}^d)$ Lemma 2.3 implies $p \le p_0$. Example 5, (2.13) and (2.14), yields

$$\| f_\ell \,|S^{t_0}_{p_0,q_0}B(\mathbb{R}^d)\| = C\, 2^{\ell d(t_0+1-\frac{1}{p_0})} \quad \text{and} \quad \| f_\ell \,|B^{td}_{p,q}(\mathbb{R}^d)\| = C\, 2^{\ell d(t+1-\frac{1}{p})}$$

with $C > 0$ independent of ℓ. The embedding

$$B^{td}_{p,q}(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0}B(\mathbb{R}^d)$$

implies

$$d\Big(t_0 + 1 - \frac{1}{p_0}\Big) \le d\Big(t + 1 - \frac{1}{p}\Big) \qquad \text{or} \qquad t_0 - \frac{1}{p_0} \le t - \frac{1}{p}\,.$$

Working with test function h_ℓ in Example 5 in the case $t_0 - \frac{1}{p_0} = t - \frac{1}{p}$ with $a_j := 2^{-jd(t+1-1/p)}$ we obtain $q \le q_0$. Hence, we conclude from Lemma 2.20 that

$$S^t_{p,q}B(\mathbb{R}^d) \hookrightarrow S^{t_0}_{p_0,q_0}B(\mathbb{R}^d).$$

Step 2. By employing Lemma 2.22 and test functions in Examples 4, 5, 6 and 7 we obtain the optimality for Triebel-Lizorkin spaces as well.

Remark 2.24. Comparing Theorem 2.19 it is natural to ask also for the optimality of $S^t_{p,q}A(\mathbb{R}^d) \hookrightarrow A^t_{p,q}(\mathbb{R}^d)$ in the other direction, i.e., we fix $S^t_{p,q}A(\mathbb{R}^d)$ and look for spaces $A^{t_0}_{p_0,q_0}(\mathbb{R}^d)$ such that (2.17) is true. For this we consider a special situation. Theorems 2.4 and 2.10 yield $S^2_{1,2}A(\mathbb{R}^d) \hookrightarrow A^2_{1,2}(\mathbb{R}^d)$. On the other hand, a Sobolev-type embedding and Theorems 2.4, 2.10 imply

$$S^2_{1,2}A(\mathbb{R}^d) \hookrightarrow S^{3/2}_{2,2}A(\mathbb{R}^d) \hookrightarrow A^{3/2}_{2,2}(\mathbb{R}^d)\,.$$

However for $d \ge 2$ these isotropic Besove-Lizorkin-Triebel spaces $A^2_{1,2}(\mathbb{R}^d)$ and $A^{3/2}_{2,2}(\mathbb{R}^d)$ are not comparable. Hence, an optimality in such a wide sense is not true.

3 Pointwise multipliers and change of variable operators

3.1 Pointwise multipliers

Let X be a Banach space of measurable functions defined on a domain $D_d \subset \mathbb{R}^d$. A functions f on D_d is called a pointwise multiplier for X if $f \cdot g \in X$ for all $g \in X$. If $X \hookrightarrow L_p(D_d)$ for some $0 < p \le \infty$, as a consequence of the Closed Graph Theorem, we obtain that the linear operator $T_f : \; g \mapsto f \cdot g$, associated to such a pointwise multiplier, must be continuous in X, i.e., $T_f \in \mathcal{L}(X)$, see [71, page 33]. By $M(X)$ we denote the set of all pointwise multipliers for X, i.e.,

$$M(X) := \big\{ f : \; f \cdot g \in X \quad \forall g \in X \big\}$$

and equip this set with the norm of the operator T_f

$$\| \, f \, | M(X) \| := \| \, T_f : \; X \to X \| = \sup_{\|g|X\| \le 1} \| \, f \cdot g \, | X \| \, .$$

We shall call X an algebra with respect to pointwise multiplication (for short a multiplication algebra) if $f \cdot g \in X$ for all $f, g \in X$ and there exist a constant $C > 0$ such that

$$\| f \cdot g \, | \, X \| \le C \| \, f \, | \, X \| \cdot \| \, g \, | \, X \|$$

holds for all $f, g \in X$. It is obvious that if X is a multiplication algebra we have, $X \hookrightarrow M(X)$.

In this section we shall describe the set of all pointwise multipliers for Sobolev spaces $S^m_p W(\mathbb{R}^d)$ and Besov spaces $S^t_{p,p} B(\mathbb{R}^d)$ of dominating mixed smoothness under certain restrictions. In addition we shall give necessary and sufficient conditions for the case that these spaces form algebras with respect to pointwise multiplication. Concerning the counterparts for the isotropic case, let us refer to Strichartz [111], Peetre [82], Triebel [129], Maz'ya, Shaposnikova [70, 71] and Runst, Sickel [100], to mention at least a few.

3.1.1 Pointwise multipliers for Sobolev spaces

One of our main results in this section reads as follows.

Theorem 3.1. *Let $m \in \mathbb{N}$ and $1 < p < \infty$. Then $S^m_p W(\mathbb{R}^d)$ is a multiplication algebra.*

Remark 3.2. Recall that if $m \in \mathbb{N}$ and $1 < p < \infty$ then the space $S^m_p W(\mathbb{R}^d)$ is continuously embedded into $C(\mathbb{R}^d)$, see Lemma 1.33. In the case $m = 0$, the space $S^0_p W(\mathbb{R}^d) = L_p(\mathbb{R}^d)$ does not form a multiplication algebra.

As a supplement we study the case $p = \infty$ by considering the spaces $C^m_{\mathrm{mix}}(\mathbb{R}^d)$ instead of $S^m_\infty W(\mathbb{R}^d)$.

Definition 3.3. Let $m \in \mathbb{N}$. Then $C^m_{\mathrm{mix}}(\mathbb{R}^d)$ is the collection of all continuous functions $f : \; \mathbb{R}^d \to \mathbb{C}$ such that all derivatives $D^\alpha f$ with $|\alpha|_\infty \le m$ are continuous as well and

$$\| \, f \, | C^m_{\mathrm{mix}}(\mathbb{R}^d) \| := \sum_{|\alpha|_\infty \le m} \; \sup_{x \in \mathbb{R}^d} \, | \, D^\alpha f(x) \, | < \infty \, .$$

Theorem 3.4. *Let $m \in \mathbb{N}$. Then $C^m_{\mathrm{mix}}(\mathbb{R}^d)$ is a multiplication algebra.*

The proof of Theorem 3.4 is almost trivial. To prepare the proof of Theorem 3.1 we need the following lemma.

Lemma 3.5. *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Let $\beta \in \mathbb{N}_0^d$ such that there exists some $L \in \mathbb{N}$, $L < d$, and $\beta = (m, \ldots, m, \beta_{L+1}, \ldots, \beta_d)$ where $\max_{j=L+1,\ldots,d} \beta_j < m$. Let $N \in \mathbb{N}$ satisfy $L \le N \le d$. Then there exists a constant C such that*

$$\Big(\int\limits_{\mathbb{R}^N} \sup_{x_{N+1},\ldots,x_d \in \mathbb{R}} \big| D^\beta f(x) \big|^p \prod_{j=1}^N \, \mathrm{d}x_j \Big)^{1/p} \le C \, \| \, f \, | S^m_p W(\mathbb{R}^d) \|$$

holds for all $f \in S^m_p W(\mathbb{R}^d)$.

Proof. If $N = d$ the assertion is obvious. We consider the case $N < d$. Using the density of functions with compactly supported Fourier transform in $S^m_p W(\mathbb{R}^d)$, see Theorem 1.27, we may assume that support of $\mathcal{F}f$ is compact. It follows that $f \in C^\infty(\mathbb{R}^d)$. Let $\{\chi_k\}_k$ be the non-smooth decomposition of unity defined in (1.10). Then we have

$$f(x) = \sum_{k \in \mathbb{N}_0^d} \mathcal{F}^{-1}[\chi_k \mathcal{F} f](x) \, , \qquad x \in \mathbb{R}^d, \tag{3.1}$$

where the sum on the right-hand side of (3.1) has only a finite number of nontrivial terms. Consequently we obtain

$$D^\beta f(x) = \sum_{k \in \mathbb{N}_0^d} \mathcal{F}^{-1}[\chi_k \mathcal{F} D^\beta f](x) \, , \qquad x \in \mathbb{R}^d.$$

Let $\mathcal{F}_n$ denote the Fourier transform on $\mathbb{R}^n$. Freezing $x_1, \ldots, x_N$ and choosing $n = d - N$ we get as above

$$D^\beta f(x) = \sum_{k_{N+1},\ldots,k_d \in \mathbb{N}_0} \mathcal{F}_n^{-1}[\chi_{k_{N+1}} \otimes \ldots \otimes \chi_{k_d} \mathcal{F}_n D^\beta f](x) \, , \qquad x \in \mathbb{R}^d.$$

By making use of this identity, triangle inequality and the Nikol'skij inequality, stated in Theorem 1.5, we conclude

$$\begin{aligned} I &:= \Big(\int\limits_{\mathbb{R}^N} \sup_{x_{N+1},\ldots,x_d \in \mathbb{R}} \big| D^\beta f(x) \big|^p \prod_{j=1}^N \, \mathrm{d}x_j \Big)^{1/p} \\ &\le \sum_{k_{N+1},\ldots,k_d \in \mathbb{N}_0} \Big(\int\limits_{\mathbb{R}^N} \sup_{x_{N+1},\ldots,x_d \in \mathbb{R}} \big| \mathcal{F}_n^{-1}[\chi_{k_{N+1}} \otimes \ldots \otimes \chi_{k_d} \mathcal{F}_n D^\beta f](x) \big|^p \prod_{j=1}^N \, \mathrm{d}x_j \Big)^{1/p} \\ &\le c_1 \sum_{k_{N+1},\ldots,k_d \in \mathbb{N}_0} \Big(\prod_{j=N+1}^d 2^{\frac{k_j}{p}} \Big) \Big(\int\limits_{\mathbb{R}^d} \big| \mathcal{F}_n^{-1}[\chi_{k_{N+1}} \otimes \ldots \otimes \chi_{k_d} \mathcal{F}_n D^\beta f](x) \big|^p \, \mathrm{d}x \Big)^{1/p} . \end{aligned}$$

The Littlewood-Paley assertion, see Theorem 1.27, implies

$$\begin{aligned} &\Big(\int\limits_{\mathbb{R}^N} \big| \mathcal{F}_n^{-1}[\chi_{k_{N+1}} \otimes \ldots \otimes \chi_{k_d} \mathcal{F}_n D^\beta f](x) \big|^p \prod_{j=1}^N \, \mathrm{d}x_j \Big)^{1/p} \\ &\qquad \le c_2 \Big\{ \int\limits_{\mathbb{R}^N} \Big(\sum_{k_1,\ldots,k_N \in \mathbb{N}_0} \big| \mathcal{F}^{-1}[\chi_{k_1} \otimes \ldots \otimes \chi_{k_d} \mathcal{F} D^\beta f](x) \big|^2 \Big)^{p/2} \prod_{j=1}^N \, \mathrm{d}x_j \Big\}^{1/p} . \end{aligned}$$

We define a multi-index $\alpha \in \mathbb{N}_0^d$ by taking $\alpha_i + \beta_i = m$ for $i = 1, \ldots, d$. Inserting this inequality in the previously obtained one we find

$$
\begin{aligned}
I \le\; & c_3 \sum_{k_{N+1},\ldots,k_d \in \mathbb{N}_0} \Big(\prod_{j=N+1}^{d} 2^{\frac{k_j}{p}} \Big) \Big\| \Big(\sum_{k_1,\ldots,k_N \in \mathbb{N}_0} \big| \mathcal{F}^{-1} \big[\chi_k \mathcal{F} D^\beta f \big] \big|^2 \Big)^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| \\
=\; & c_3 \sum_{k_{N+1},\ldots,k_d \in \mathbb{N}_0} \Big(\prod_{j=N+1}^{d} 2^{k_j(\frac{1}{p} - \alpha_j)} \Big) \\
& \times \quad \Big\| \Big\{ \sum_{k_1,\ldots,k_N \in \mathbb{N}_0} \Big(\prod_{j=N+1}^{d} 2^{2k_j\alpha_j} \Big) \big| \mathcal{F}^{-1} \big[\chi_k \mathcal{F} D^\beta f \big] \big|^2 \Big\}^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| \\
\le\; & c_4 \Big\| \Big\{ \sum_{k \in \mathbb{N}_0^d} \Big(\prod_{j=N+1}^{d} 2^{2k_j\alpha_j} \Big) \big| \mathcal{F}^{-1} \big[\chi_k \mathcal{F} D^\beta f \big] \big|^2 \Big\}^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| ,
\end{aligned}
$$

where we used $\alpha_j \ge 1 > \frac{1}{p}$, $j = N+1, \ldots, d$. Let $\phi_0, \phi \in C_0^\infty(\mathbb{R})$ be functions such that

$$
\phi_0 \equiv 1 \quad \text{on} \quad [-1,1] \qquad \text{and} \qquad \phi \equiv 1 \quad \text{on} \quad \operatorname{supp}(\chi_1) \, .
$$

For $j \in \mathbb{N}$ we put $\phi_j(t) := \phi(2^{-j+1}t)$ and $\phi_k := \phi_{k_1} \otimes \ldots \otimes \phi_{k_d}$ if $k \in \mathbb{N}_0^d$. Then it follows

$$
\begin{aligned}
& \Big\| \Big\{ \sum_{k \in \mathbb{N}_0^d} \Big(\prod_{j=N+1}^{d} 2^{2k_j\alpha_j} \Big) \big| \mathcal{F}^{-1} \big[\chi_k(\xi) \, \phi_k(\xi) \, \xi^\beta \, \mathcal{F} f(\xi) \big] (\,\cdot\,) \big|^2 \Big\}^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| \\
& \qquad = \Big\| \Big\{ \sum_{k \in \mathbb{N}_0^d} 2^{2|k|_1 m} \, \big| \mathcal{F}^{-1} \big[M_k \, \chi_k \, \mathcal{F} f \big] \big|^2 \Big\}^{1/2} \Big| L_p(\mathbb{R}^d) \Big\| ,
\end{aligned}
$$

where $\xi \in \mathbb{R}^d$ and

$$
M_k(\xi) := \phi_k(\xi) \Big(\prod_{j=1}^{N} 2^{-k_j m} \xi_j^{\beta_j} \Big) \Big(\prod_{j=N+1}^{d} 2^{k_j(\alpha_j - m)} \xi_j^{\beta_j} \Big) .
$$

Observe that in case $k_j \ge 1$ for $j = 1, \ldots, d$, we have

$$
\begin{aligned}
& \| \, M_k(2^k \diamond \cdot\,) \, | S_p^r W(\mathbb{R}^d) \| \\
& \qquad = \Big(\prod_{j=1}^{N} 2^{k_j(\beta_j - m)} \prod_{j=N+1}^{d} 2^{k_j(\alpha_j + \beta_j - m)} \Big) \| \, \phi_{\bar{1}}(2\xi) \, \xi^\beta \, | S_p^r W(\mathbb{R}^d) \| < \infty
\end{aligned}
$$

for any $r > 0$. For the remaining k a more or less obvious modification can be applied. Hence we find

$$
\sup_{k \in \mathbb{N}_0^d} \| \, M_k(2^k \diamond \cdot\,) \, | S_p^r W(\mathbb{R}^d) \| < \infty
$$

if

$$
|\beta|_\infty \le m \qquad \text{and} \qquad \alpha_j + \beta_j \le m \, , \quad j = N+1, \ldots, d \, .
$$

But this is guaranteed by our assumptions. Now Lemma 1.11 yields

$$
I \;\le\; c_5 \Big\| \Big(\sum_{k \in \mathbb{N}_0^d} 2^{2|k|_1 m} \big| \mathcal{F}^{-1} \chi_k \mathcal{F} f \big|^2 \Big)^{1/2} \Big| L_p(\mathbb{R}^d) \Big\|
$$

which completes the proof. ■

Proof of Theorem 3.1. Let $f, g \in S^m_p W(\mathbb{R}^d)$. By definition we have

$$\| f \cdot g | S^m_p W(\mathbb{R}^d) \| = \sum_{\alpha \in \mathbb{N}_0^d,\ |\alpha|_\infty \le m} \| D^\alpha (f \cdot g) | L_p(\mathbb{R}^d) \| .$$

Using the density of functions with compactly supported Fourier transform in $S^m_p W(\mathbb{R}^d)$ we may assume that $\mathcal{F} f$ and $\mathcal{F} g$ have compact supports. Hence f and g are C^∞ functions. Leibniz rule yields

$$D^\alpha (f \cdot g)(x) = \sum_{\beta \in \mathbb{N}_0^d:\ 0 \le \beta \le \alpha} \binom{\alpha}{\beta} D^\beta f(x)\, D^{\alpha - \beta} g(x) \, .$$

Let us assume $|\beta|_\infty < m$. Then from the definition of $S^m_p W(\mathbb{R}^d)$ we derive $D^\beta f \in S_p^{m - |\beta|_\infty} W(\mathbb{R}^d)$ and Lemma 1.33 we conclude $S_p^{m-|\beta|_\infty} W(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$. Hence

$$\begin{aligned} \| D^\beta f \, D^{\alpha-\beta} g \, | L_p(\mathbb{R}^d) \| & \le \| D^\beta f \, | C(\mathbb{R}^d) \| \cdot \| D^{\alpha - \beta} g \, | L_p(\mathbb{R}^d) \| \\ & \le c_1 \, \| D^\beta f \, | S_p^{m - |\beta|_\infty} W(\mathbb{R}^d) \| \cdot \| \, g \, | S^m_p W(\mathbb{R}^d) \| \\ & \le c_1 \, \| \, f \, | S^m_p W(\mathbb{R}^d) \| \cdot \| \, g \, | S^m_p W(\mathbb{R}^d) \| \, , \end{aligned}$$

where $c_1 := \| id : \ S_p^{m - |\beta|} W(\mathbb{R}^d) \to C(\mathbb{R}^d) \|$. Of course, a similar argument can be applied if $|\alpha - \beta|_\infty < m$. It remains to deal with the situation $|\beta|_\infty = |\alpha - \beta|_\infty = m$. Without loss of generality we assume

$$\beta = (m, \ldots, m, \beta_{L+1}, \ldots, \beta_N, 0, \ldots, 0)$$

for some $L, N \in \mathbb{N}$ and

$$0 < \beta_j < m \, , \qquad L + 1 \le j \le N < d \, .$$

But now we can use Lemma 3.5 and obtain

$$\begin{aligned} & \| D^\beta f \cdot D^{\alpha - \beta} g \, | L_p(\mathbb{R}^d) \| \\ & \le \Big(\int_{\mathbb{R}^N} \sup_{x_{N+1}, \ldots, x_d \in \mathbb{R}} | D^\beta f(x)|^p \prod_{j=1}^N \mathrm{d} x_j \Big)^{1/p} \Big(\int_{\mathbb{R}^{d-N}} \sup_{x_1, \ldots, x_N \in \mathbb{R}} | D^{\alpha-\beta} g(x)|^p \prod_{j=N+1}^d \mathrm{d} x_j \Big)^{1/p} \\ & \le \ C^2 \, \| \, f \, | S^m_p W(\mathbb{R}^d) \| \cdot \| \, g \, | S^m_p W(\mathbb{R}^d) \| \end{aligned}$$

which proves the claim. ■

Let ψ be a non-negative $C_0^\infty(\mathbb{R}^d)$ function. We put $\psi_\mu(x) = \psi(x - \mu)$, $\mu \in \mathbb{Z}^d$, $x \in \mathbb{R}^d$ and assume that

$$\sum_{\mu \in \mathbb{Z}^d} \psi_\mu(x) = 1 \qquad \text{for all } x \in \mathbb{R}^d \, . \tag{3.2}$$

Definition 3.6. Let the Banach space X be continuously embedded into $L_1^{\ell oc}(\mathbb{R}^d)$.
(i) $X^{\ell oc}$ is the collection of all $g \in L_1^{\ell oc}(\mathbb{R}^d)$ such that $\varphi \cdot g \in X$ for all test functions $\varphi \in C_0^\infty(\mathbb{R}^d)$.
(ii) Let $\psi \in C_0^\infty(\mathbb{R}^d)$ satisfy (3.2). Then X_{unif} is the collection of all $f \in X^{\ell oc}$ such that

$$\| \, f \, | X_{\mathrm{unif}} \|_\psi = \sup_{\mu \in \mathbb{Z}^d} \| \, \psi_\mu \cdot f \, | X \| < \infty .$$

Remark 3.7. The space $S^m_pW(\mathbb{R}^d)_{\text{unif}}$ is independent of the special choice of ψ (in the sense of equivalent norms). This is a consequence of Theorem 3.1.

Based on Theorems 3.1 and 3.4, we are now in position to describe the spaces $M(S^m_pW(\mathbb{R}^d))$ and $M(C^m_{\text{mix}}(\mathbb{R}^d))$.

Theorem 3.8. (i) *Let* $1 < p < \infty$ *and* $m \in \mathbb{N}$. *Then we have*

$$M(S^m_pW(\mathbb{R}^d)) = S^m_pW(\mathbb{R}^d)_{\text{unif}}$$

in the sense of equivalent norms.
(ii) *We have*

$$M(C^m_{\text{mix}}(\mathbb{R}^d)) = C^m_{\text{mix}}(\mathbb{R}^d)$$

in the sense of equivalent norms.

Proof. *Step 1.* We first prove the localization property of the spaces $S^m_pW(\mathbb{R}^d)$, i.e.,

$$\|f|S^m_pW(\mathbb{R}^d)\| \asymp \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S^m_pW(\mathbb{R}^d)\|^p\Big)^{1/p} \tag{3.3}$$

holds for all $f \in S^m_pW(\mathbb{R}^d)$. This is a consequence of the localization property of $L_p(\mathbb{R}^d)$

$$\|f|L_p(\mathbb{R}^d)\| \asymp \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|L_p(\mathbb{R}^d)\|^p\Big)^{1/p}$$

for all $f \in L_p(\mathbb{R}^d)$ with $1 < p < \infty$, see Strichartz [111]. Indeed, from this we obtain

$$\|f|S^m_pW(\mathbb{R}^d)\| \asymp \Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \|\psi_\mu D^\alpha f|L_p(\mathbb{R}^d)\|^p\Big)^{1/p} \tag{3.4}$$

and

$$\Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S^m_pW(\mathbb{R}^d)\|^p\Big)^{1/p} \asymp \Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \|D^\alpha(\psi_\mu f)|L_p(\mathbb{R}^d)\|^p\Big)^{1/p}. \tag{3.5}$$

For $\mu \in \mathbb{Z}^d$ we denote $\Delta_\mu = \{\nu \in \mathbb{Z}^d : \operatorname{supp}\psi_\nu \cap \operatorname{supp}\psi_\mu \neq \emptyset\}$. It follows that $|\Delta_\mu| \le c$ where c independent of μ. Taking this into account and $\psi \in C^\infty_0(\mathbb{R}^d)$ we have

$$\begin{aligned}
\Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \|\psi_\mu D^\alpha f|L_p(\mathbb{R}^d)\|^p\Big)^{1/p} &= \Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \Big\|\psi_\mu D^\alpha\Big(\sum_{\nu\in\Delta_\mu}\psi_\nu f\Big)\Big|L_p(\mathbb{R}^d)\Big\|^p\Big)^{1/p}\\
&\lesssim \Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \|D^\alpha(\psi_\mu f)|L_p(\mathbb{R}^d)\|^p\Big)^{1/p}\\
&= \Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \Big\|\sum_{\nu\in\Delta_\mu}\psi_\nu D^\alpha(\psi_\mu f)\Big|L_p(\mathbb{R}^d)\Big\|^p\Big)^{1/p}\\
&\lesssim \Big(\sum_{\mu\in\mathbb{Z}^d}\sum_{|\alpha|_\infty\le m} \|\psi_\mu D^\alpha f|L_p(\mathbb{R}^d)\|^p\Big)^{1/p}.
\end{aligned}$$

This together with (3.4) and (3.5) implies (3.3).
Step 2. Proof of (i). Let $\phi \in C_0^\infty(\mathbb{R}^d)$ such that $\phi \equiv 1$ on support of ψ. Let $f \in S_p^mW(\mathbb{R}^d)$ and $g \in S_p^mW(\mathbb{R}^d)_{\mathrm{unif}}$. Employing the localization principle and Theorem 3.1 we obtain

$$\begin{aligned}\|f \cdot g|S_p^mW(\mathbb{R}^d)\| &\leq c_1 \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu\phi_\mu g f|S_p^mW(\mathbb{R}^d)\|^p\Big)^{1/p} \\ &\leq c_2 \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S_p^mW(\mathbb{R}^d)\|^p \cdot \|\phi_\mu g|S_p^mW(\mathbb{R}^d)\|^p\Big)^{1/p} \\ &\leq c_3 \|f|S_p^mW(\mathbb{R}^d)\| \cdot \sup_{\mu\in\mathbb{Z}^d} \|\phi_\mu g|S_p^mW(\mathbb{R}^d)\|.\end{aligned}$$

Since cardinality of the set $D_\mu := \{\nu \in \mathbb{Z}^d : \operatorname{supp}\phi_\mu \cap \psi_\nu \neq \emptyset\}$ is finite and independent of μ, from Theorem 3.1 we obtain

$$\|\phi_\mu g|S_p^mW(\mathbb{R}^d)\| = \Big\|\phi_\mu g\Big(\sum_{\nu\in D_\mu} \psi_\nu\Big)\Big|S_p^mW(\mathbb{R}^d)\Big\| \leq c \sup_{\nu\in\mathbb{Z}^d} \|\psi_\nu g|S_p^mW(\mathbb{R}^d)\|$$

which implies

$$\|f \cdot g|S_p^mW(\mathbb{R}^d)\| \leq c_4 \|f|S_p^mW(\mathbb{R}^d)\| \cdot \sup_{\mu\in\mathbb{Z}^d} \|\psi_\mu g|S_p^mW(\mathbb{R}^d)\|.$$

Hence,

$$S_p^mW(\mathbb{R}^d)_{\mathrm{unif}} \hookrightarrow M(S_p^mW(\mathbb{R}^d)).$$

On the other hand, with $g \in M(S_p^mW(\mathbb{R}^d))$, we derive

$$\begin{aligned}\|\psi_\mu g|S_p^mW(\mathbb{R}^d)\| &\leq \|g|M(S_p^mW(\mathbb{R}^d))\| \cdot \|\psi_\mu|S_p^mW(\mathbb{R}^d)\| \\ &= \|g|M(S_p^mW(\mathbb{R}^d))\| \cdot \|\psi|S_p^mW(\mathbb{R}^d)\|.\end{aligned}$$

Consequently

$$M(S_p^mW(\mathbb{R}^d)) \hookrightarrow S_p^mW(\mathbb{R}^d)_{\mathrm{unif}}$$

which proves (i).
Step 3. Proof of (ii). From algebra property of $C^m_{\mathrm{mix}}(\mathbb{R}^d)$, see Theorem 3.4, we obtain immediately $C^m_{\mathrm{mix}}(\mathbb{R}^d) \hookrightarrow M(C^m_{\mathrm{mix}}(\mathbb{R}^d))$. Assume that $f \in M(C^m_{\mathrm{mix}}(\mathbb{R}^d))$. By definition of pointwise multiplier we have

$$\|f \cdot g|C^m_{\mathrm{mix}}(\mathbb{R}^d)\| \leq c\, \|f|M(C^m_{\mathrm{mix}}(\mathbb{R}^d))\| \cdot \|g|C^m_{\mathrm{mix}}(\mathbb{R}^d)\|$$

for all $g \in C^m_{\mathrm{mix}}(\mathbb{R}^d)$. But the function $g \equiv 1 \in C^m_{\mathrm{mix}}(\mathbb{R}^d)$. This implies $f \in C^m_{\mathrm{mix}}(\mathbb{R}^d)$. The proof is complete. ■

Theorem 3.9. *Let $d > 1$ and $m \in \mathbb{N}$.*
(i) *Then there exists no constant $C > 0$ such that*

$$\|f \cdot g\,|C^m_{\mathrm{mix}}(\mathbb{R}^d)\| \leq C\,\big(\|\,f\,|C^m_{\mathrm{mix}}(\mathbb{R}^d)\| \cdot \|\,g\,|L_\infty(\mathbb{R}^d)\| + \|\,f\,|L_\infty(\mathbb{R}^d)\| \cdot \|g|C^m_{\mathrm{mix}}(\mathbb{R}^d)\|\big)$$

holds for all $f, g \in C^m_{\mathrm{mix}}(\mathbb{R}^d)$.
(ii) *Let $1 < p < \infty$. There exists no constant $C > 0$ such that*

$$\|f \cdot g|S_p^mW(\mathbb{R}^d)\| \leq C\big(\|f|S_p^mW(\mathbb{R}^d)\| \cdot \|g|L_\infty(\mathbb{R}^d)\| + \|f|L_\infty(\mathbb{R}^d)\| \cdot \|g|S_p^mW(\mathbb{R}^d)\|\big)$$

holds for all $f, g \in S_p^mW(\mathbb{R}^d)$.

Proof. Here we can work with the same test functions as in proof of Theorem 3.17 below. Since the B-case is a bit more complicated we give details in this situation. ■

3.1.2 Pointwise multipliers for Besov spaces

In this section we shall employ the characterization by differences to prove the algebra property of $S^t_{p,p}B(\mathbb{R}^d)$ as in the classical paper [111] of Strichartz or in the monographs [70, 71] by Maz'ya and Shaposnikova. It seems that the method of using paraproducts, already applied in Peetre [82], Triebel [129, 130] or Runst, Sickel [100], is less convenient in the context of dominating mixed smoothness. The main result with respect to Besov spaces of dominating mixed smoothness reads as follows.

Theorem 3.10. *Let $1 \le p \le \infty$ and $t > 0$. Then $S^t_{p,p}B(\mathbb{R}^d)$ is an algebra if and only if either $t > 1/p$ or $t = p = 1$.*

A very useful relation between Peetre maximal function and differences is given by the following lemma, see Ullrich [137] and Schmeißer, Triebel [104, Section 2.3.3].

Lemma 3.11. *Let $a > 0$ and $m \in \mathbb{N}$. Then there exists a constant C such that*

$$|\Delta^m_h f(\xi)| \le C \, \max\{1, |bh|^a\} \, \min\{1, |bh|^m\} \, P_{b,a}f(\xi) \,.$$

holds for all $b > 0$, all $h \neq 0$, all $\xi \in \mathbb{R}$ and all $f \in \mathcal{S}'(\mathbb{R})$ satisfying $\operatorname{supp}(\mathcal{F}f) \subset [-b, b]$.

Applying the above result iteratively with respect to the components in $e \subset [d]$ we get the following modified version in multivariate situation.

Lemma 3.12. *Let $a > 0$, $e \subset [d]$, $m \in \mathbb{N}^d_0$ and $h \in \mathbb{R}^d$. Let further $f \in \mathcal{S}'(\mathbb{R}^d)$ with* $\operatorname{supp}(\mathcal{F}f) \subset Q_b$, *where*

$$Q_b := [-b_1, b_1] \times \ldots \times [-b_d, b_d] \,, \quad b_i > 0, \quad i = 1, \ldots, d.$$

Then there exists a constant $C > 0$ (independent of f, b, x and h) such that

$$|\Delta^{m,e}_h f(x)| \le C \left(\prod_{i \in e} \max\{1, |b_i h_i|^a\} \, \min\{1, |b_i h_i|^{m_i}\} \right) P_{b,a}f(x)$$

holds for all $x \in \mathbb{R}^d$.

Proof of Theorem 3.10. We divide the proof into two parts.
Part I - sufficiency. *Step 1.* Let $t < m \le t + 1$. Since the norm $\| \cdot \mid S^t_{p,p}B(\mathbb{R}^d)\|^{(m)}$ does not depend on $m > t$ in the sense of equivalent norms, we shall prove that

$$\|f \cdot g \mid S^t_{p,p}B(\mathbb{R}^d)\|^{(2m)} \le C \, \|f|S^t_{p,p}B(\mathbb{R}^d)\| \cdot \|g|S^t_{p,p}B(\mathbb{R}^d)\|$$

holds for all $f, g \in S^t_{p,p}B(\mathbb{R}^d)$. Taking into account Lemma 1.33 (ii) we obtain

$$\|f \cdot g \,|L_p(\mathbb{R}^d)\| \;\le\; \|f|L_p(\mathbb{R}^d)\| \cdot \|g|C(\mathbb{R}^d)\| \;\le\; \|f|S^t_{p,p}B(\mathbb{R}^d)\| \cdot \|g|S^t_{p,p}B(\mathbb{R}^d)\|.$$

This inequality can be interpreted as the estimate needed for the term with $e = \emptyset$. Next we need some identities for differences. Note that if ψ, $\phi :\; \mathbb{R} \to \mathbb{C}$ and $m \in \mathbb{N}$ we have

$$\Delta^m_h(\psi\phi)(\xi) = \sum_{j=0}^{m} \binom{m}{j} \Delta^{m-j}_h \psi(\xi + jh) \, \Delta^j_h \phi(\xi), \qquad \xi, h \in \mathbb{R} \,, \tag{3.6}$$

which can be proved by induction on m, see also Triebel [131, page 197]. Let $e \subset [d]$, $e \neq \emptyset$. Then we derive from (3.6) that

$$\Delta_h^{2\bar{m},e}(f \cdot g)(x) = \sum_{u \in \mathbb{N}_0^d(e),\, |u|_\infty \le 2m} \binom{2\bar{m}}{u} \Delta_h^{2\bar{m}-u,e} f(x + u \diamond h) \Delta_h^{u,e} g(x)\,, \qquad x, h \in \mathbb{R}^d\,, \quad (3.7)$$

holds. Here $2\bar{m} - u := (2m - u_1, \ldots, 2m - u_d)$ and

$$\binom{2\bar{m}}{u} = \prod_{i \in e} \binom{2m}{u_i}\,.$$

The main step of the proof will consists in estimating the terms

$$S_{e,u} := \Big\{ \sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 p} \Big(\sup_{|h_i| < 2^{-k_i}, i \in e} \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \Delta_h^{u,e} g(\cdot) \big| L_p(\mathbb{R}^d) \big\| \Big)^p \Big\}^{1/p}$$

$e \neq \emptyset$, $u \in \mathbb{N}_0^d(e)$, $|u|_\infty \le 2m$, by considering some different cases.
Step 2. The case $u_i \le m$ for all $i \in e$. Obviously we have $2m - u_i \ge m,\ i \in e$. Using a change of variables in the L_p-integral we obtain

$$\begin{aligned} \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \Delta_h^{u,e} g(\cdot) \big| L_p(\mathbb{R}^d) \big\| &\le \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \big| L_p(\mathbb{R}^d) \big\| \sup_{x \in \mathbb{R}^d} |\Delta_h^{u,e} g(x)| \\ &\le c_1 \|g|C(\mathbb{R}^d)\| \cdot \big\| \Delta_h^{\bar{m},e} f(\cdot) \big| L_p(\mathbb{R}^d) \big\|\,. \end{aligned}$$

The embedding $S^t_{p,p}B(\mathbb{R}^d) \hookrightarrow C(\mathbb{R}^d)$ implies

$$\begin{aligned} \sup_{|h_i| < 2^{-k_i}, i \in e} \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \Delta_h^{u,e} g(\cdot) \,\big| L_p(\mathbb{R}^d) \big\| &\le c_1 \, \|g|C(\mathbb{R}^d)\| \, \omega^e_{\bar{m}}(f, 2^{-k})_p \\ &\le c_2 \, \|g|S^t_{p,p}B(\mathbb{R}^d)\| \, \omega^e_{\bar{m}}(f, 2^{-k})_p\,. \end{aligned}$$

Consequently we have

$$\begin{aligned} S_{e,u} \le\ & c_2 \, \|g|S^t_{p,p}B(\mathbb{R}^d)\| \Big(\sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 p} \, \omega^e_{\bar{m}}(f, 2^{-k})_p^p \Big)^{1/p} \\ \le\ & c_2 \, \|g|S^t_{p,p}B(\mathbb{R}^d)\| \cdot \|f|S^t_{p,p}B(\mathbb{R}^d)\|\,. \end{aligned}$$

The case $u_i \ge m$ for all $i \in e$ can be handled in the same way by interchanging the roles of f and g.
Step 3. The remaining cases. Let there exist $L, N \in \mathbb{N}$ and $L < N \le d$ such that $e = \{1, 2, \ldots, N\}$, $u \in \mathbb{N}_0^d(e)$ and

$$u := (u_1, \ldots, u_L, u_{L+1}, \ldots, u_N, 0, \ldots, 0)$$

with

$$m \le u_i \le 2m\,, \quad i = 1, \ldots, L, \qquad 0 \le u_i < m, \quad i = L+1, \ldots, N.$$

By assuming $|u|_\infty > m$ we cover all remaining cases up to an enumeration.
Substep 3.1. Let $t > 1/p$. Working with the tensor product system $\{\varphi_k\}_{k \in \mathbb{N}_0^d}$, see Section 1.2, we conclude

$$f = \sum_{\ell \in \mathbb{Z}^d} \mathcal{F}^{-1}[\varphi_{k+\ell} \mathcal{F} f] \qquad (3.8)$$

with convergence in $S^t_{p,p}B(\mathbb{R}^d)$ and therefore in $C(\mathbb{R}^d)$. Here we used the convention that $\varphi_k \equiv 0$ if $\min_{i=1,\ldots,d} k_i < 0$. Hence we have the decompositions

$$f(x) = \sum_{\ell\in\mathbb{Z}^d} \mathcal{F}^{-1}[\varphi_{k+\ell}\mathcal{F}f](x) \qquad \text{and} \qquad g(x) = \sum_{\nu\in\mathbb{Z}^d} \mathcal{F}^{-1}[\varphi_{k+\nu}\mathcal{F}g](x)\,, \qquad x\in\mathbb{R}^d\,,$$

with convergence in $C(\mathbb{R}^d)$. To simplify notation we put

$$f_\ell := \mathcal{F}^{-1}[\varphi_\ell\mathcal{F}f] \qquad \text{and} \qquad g_\ell := \mathcal{F}^{-1}[\varphi_\ell\mathcal{F}g]\,, \qquad \ell\in\mathbb{Z}^d\,.$$

Then we obtain from triangle inequality

$$\begin{aligned}\big\|\Delta_h^{2\bar{m}-u,e} f(\cdot + u\diamond h)\Delta_h^{u,e} g(\cdot)\,\big|L_p(\mathbb{R}^d)\big\| \\ \leq \sum_{\ell,\nu\in\mathbb{Z}^d} \big\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h)\Delta_h^{u,e} g_{k+\nu}(\cdot)\,\big|L_p(\mathbb{R}^d)\big\|\,.\end{aligned}$$

We will estimate the sum on the right-hand side term by term. It follows

$$\begin{aligned}&\big\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h)\Delta_h^{u,e} g_{k+\nu}(\cdot)\,|L_p(\mathbb{R}^d)\big\| \\ &\leq \bigg(\int_{\mathbb{R}^{d-L}} \sup_{\substack{x_i\in\mathbb{R}\\ i\leq L}} \big|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(x+u\diamond h)\big|^p \prod_{i=L+1}^d \mathrm{d}x_i\bigg)^{1/p} \bigg(\int_{\mathbb{R}^L} \sup_{\substack{x_i\in\mathbb{R}\\ i>L}} \big|\Delta_h^{u,e} g_{k+\nu}(x)\big|^p \prod_{i=1}^L \mathrm{d}x_i\bigg)^{1/p}\end{aligned}$$

Let $\mathcal{F}_L$ denote the Fourier transform with respect to $(x_1,\ldots,x_L)$. Observe that for any $h\in\mathbb{R}^L$

$$\operatorname{supp}\mathcal{F}_L\big(f_{k+\ell}(\,\cdot\,+h, x_{L+1},\ldots,x_d)\big) \subset \big\{(\xi_1,\ldots,\xi_L):\ |\xi_j|\leq 3\cdot 2^{k_j+\ell_j-1}\,,\ j=1,\ldots,L\big\}\,,$$

independent of $x_{L+1},\ldots,x_d$. Consequently, Nikol'skij inequality in Theorem 1.5 yields

$$\begin{aligned}\bigg(\int_{\mathbb{R}^{d-L}} \sup_{\substack{x_i\in\mathbb{R}\\ i\leq L}} \big|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(x+u\diamond h)\big|^p \prod_{i=L+1}^d \mathrm{d}x_i\bigg)^{1/p} \\ \leq\ c_3 \bigg(\prod_{i=1}^L 2^{\frac{k_i+\ell_i}{p}}\bigg)\bigg(\int_{\mathbb{R}^d} \big|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(x+u\diamond h)\big|^p\,\mathrm{d}x\bigg)^{1/p}\end{aligned}$$

with a constant c_3 independent of f, k and ℓ. A simple change of coordinates and an analogous argument with respect to $g_{k+\nu}$ results in

$$\begin{aligned}&\big\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h)\Delta_h^{u,e} g_{k+\nu}(\cdot)\,\big|L_p(\mathbb{R}^d)\big\| \\ &\leq\ c_4 \bigg(\prod_{i=1}^L 2^{\frac{k_i+\ell_i}{p}}\bigg)\bigg(\prod_{i=L+1}^d 2^{\frac{k_i+\nu_i}{p}}\bigg)\big\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}\big|L_p(\mathbb{R}^d)\big\|\,\big\|\Delta_h^{u,e} g_{k+\nu}\big|L_p(\mathbb{R}^d)\big\|.\end{aligned}$$

We need one more notation. We put

$$\omega(\ell) := \{i\in\{1,\ldots,d\}:\ \ell_i<0\} \qquad \text{and} \qquad \overline{\omega}(\ell) := \{i\in\{1,\ldots,d\}:\ \ell_i\geq 0\}\,. \tag{3.9}$$

By writing $\Delta_h^{2\bar{m}-u,e}$ as

$$\Delta_h^{2\bar{m}-u,e} = \bigg(\prod_{i\in\overline{\omega}(\ell)\cap e} \Delta_{h_i}^{2m-u_i}\bigg)\bigg(\prod_{i\in\omega(\ell)\cap e} \Delta_{h_i}^{2m-u_i}\bigg)$$

it is easily seen that

$$\sup_{|h_i|<2^{-k_i},\, i\in e} \left\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}\middle|L_p(\mathbb{R}^d)\right\| \le c_5 \Big(\prod_{i\in\omega(\ell)\cap e} 2^{\ell_i(2m-u_i)}\Big) \left\|f_{k+\ell}\middle|L_p(\mathbb{R}^d)\right\|,$$

where we have applied Lemma 3.12 and scalar version of Theorem 1.10. Altogether we have found the estimate

$$\begin{aligned}
&\sup_{|h_i|<2^{-k_i},\, i\in e} \left\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h)\Delta_h^{u,e} g_{k+\nu}(\cdot)\,|L_p(\mathbb{R}^d)\right\| \\
&\qquad \le c_6 \Big(\prod_{i=1}^{L} 2^{\frac{k_i+\ell_i}{p}}\Big)\Big(\prod_{i=L+1}^{d} 2^{\frac{k_i+\nu_i}{p}}\Big)\Big(\prod_{i\in\omega(\ell)\cap e} 2^{\ell_i(2m-u_i)}\Big)\Big(\prod_{i\in\omega(\nu)\cap e} 2^{\nu_i u_i}\Big) \\
&\qquad\quad \times\ \|f_{k+\ell}|L_p(\mathbb{R}^d)\|\cdot\|g_{k+\nu}|L_p(\mathbb{R}^d)\|
\end{aligned}$$

with a constant c_6 independent of f, g, k, ℓ and ν. Observe that

$$\begin{aligned}
&2^{t|k|_1}\Big(\prod_{i=1}^{L} 2^{\frac{k_i+\ell_i}{p}}\Big)\Big(\prod_{i=L+1}^{d} 2^{\frac{k_i+\nu_i}{p}}\Big)\Big(\prod_{i\in\omega(\ell)\cap e} 2^{\ell_i(2m-u_i)}\Big)\Big(\prod_{i\in\omega(\nu)\cap e} 2^{\nu_i u_i}\Big) \\
&\quad = \Big(\prod_{i=1}^{d} 2^{(k_i+\ell_i)t}\, 2^{(k_i+\nu_i)t}\Big)\Big(\prod_{i=1}^{L} 2^{(k_i+\ell_i)(\frac{1}{p}-t)}\Big)\Big(\prod_{i=L+1}^{d} 2^{(k_i+\nu_i)(\frac{1}{p}-t)}\Big)\Big(\prod_{i=1}^{L} 2^{-\nu_i r}\Big) \\
&\qquad \times\ \Big(\prod_{i=L+1}^{N} 2^{-\ell_i t}\Big)\Big(\prod_{i=N+1}^{d} 2^{-\ell_i t}\Big)\Big(\prod_{i\in\omega(\ell)\cap e} 2^{\ell_i(2m-u_i)}\Big)\Big(\prod_{i\in\omega(\nu)\cap e} 2^{\nu_i u_i}\Big).
\end{aligned}$$

Later on we will have to sum up only with respect to those terms where $\min_j(k_j+\ell_j)\ge 0$ or $\min_j(k_j+\nu_j)\ge 0$. Observe that $k\in\mathbb{N}_0^d(e)$, i.e., $k_{N+1}=\ldots=k_d=0$ and therefore $\ell_{N+1},\ldots,\ell_d\ge 0$ and $\nu_{N+1},\ldots,\nu_d\ge 0$. Taking this into account it is obvious that

$$\begin{aligned}
&\Big(\prod_{i=N+1}^{d} 2^{-\ell_i t}\Big)\Big(\prod_{i=1}^{L} 2^{(k_i+\ell_i)(\frac{1}{p}-t)}\Big)\Big(\prod_{i=L+1}^{d} 2^{(k_i+\nu_i)(\frac{1}{p}-t)}\Big) \\
&\qquad \le \Big(\prod_{i=N+1}^{d} 2^{-(\ell_i+k_i)\varepsilon}\Big)\Big(\prod_{i=1}^{L} 2^{-(k_i+\ell_i)\varepsilon}\Big)\Big(\prod_{i=L+1}^{d} 2^{-(k_i+\nu_i)\varepsilon}\Big) \le 1
\end{aligned}$$

if $0\le\varepsilon\le\min(t, t-1/p)$. Let $\delta := \min(t, m-t)$. Clearly $\delta\in(0,1)$. Furthermore

$$\begin{aligned}
\Big(\prod_{i=L+1}^{N} 2^{-\ell_i t}\Big)\Big(\prod_{i\in\omega(\ell)\cap e} 2^{\ell_i(2m-u_i)}\Big) &= \Big(\prod_{\substack{L<i\le N\\ i\in\bar{\omega}(\ell)}} 2^{-\ell_i t} \prod_{\substack{L<i\le N\\ i\in\omega(\ell)}} 2^{\ell_i(2m-u_i-t)}\Big)\Big(\prod_{\substack{1\le i\le L\\ i\in\omega(\ell)}} 2^{\ell_i(2m-u_i)}\Big) \\
&\le \Big(\prod_{i=L+1}^{N} 2^{-|\ell_i|\delta}\Big)
\end{aligned}$$

and

$$\begin{aligned}
\Big(\prod_{i=1}^{L} 2^{-\nu_i r}\Big)\Big(\prod_{i\in\omega(\nu)\cap e} 2^{\nu_i u_i}\Big) &= \Big(\prod_{\substack{L<i\le N\\ i\in\omega(\nu)}} 2^{\nu_i u_i}\Big)\Big(\prod_{\substack{i\le L\\ i\in\omega(\nu)}} 2^{\nu_i(u_i-t)} \prod_{\substack{i\le L\\ i\in\bar{\omega}(\nu)}} 2^{-\nu_i r}\Big) \\
&\le \Big(\prod_{i=1}^{L} 2^{-|\nu_i|\delta}\Big).
\end{aligned}$$

Consequently we obtain

$$
\begin{aligned}
& 2^{t|k|_1} \sup_{|h_i|<2^{-k_i}\ ,i\in e} \left\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h)\Delta_h^{u,e} g_{k+\nu}(\cdot)\,|L_p(\mathbb{R}^d)\right\| \\
& \le c_6\Big(\prod_{i=N+1}^{d} 2^{-(\ell_i+k_i)\varepsilon}\Big)\Big(\prod_{i=1}^{L} 2^{-(k_i+\ell_i)\varepsilon}\Big)\Big(\prod_{i=L+1}^{d} 2^{-(k_i+\nu_i)\varepsilon}\Big) \\
& \times\ \Big(\prod_{i=L+1}^{N} 2^{-|\ell_i|\delta}\Big)\Big(\prod_{i=1}^{L} 2^{-|\nu_i|\delta}\Big)\Big(\prod_{i=1}^{d} 2^{(k_i+\ell_i)t}\, 2^{(k_i+\nu_i)t}\Big)\|f_{k+\ell}|L_p(\mathbb{R}^d)\|\,\|g_{k+\nu}|L_p(\mathbb{R}^d)\|.
\end{aligned}
\tag{3.10}
$$

Next we apply the inequality

$$
\sum_{j\in\mathbb{N}_0} |a_j| \le c_7\Big(\sum_{j\in\mathbb{N}_0} 2^{j\varepsilon p}|a_j|^p\Big)^{1/p},
$$

valid for all $\varepsilon>0$ with an appropriate constant c_7 depending on ε. This yields

$$
\begin{aligned}
& \Bigg\{\sum_{k\in\mathbb{N}_0^d(e)}\Bigg[2^{t|k|_1}\sum_{\substack{\ell_i\in\mathbb{Z},i\notin\{L+1,\ldots,N\}\\ \nu_i\in\mathbb{Z},L+1\le i\le d}}\ \sup_{|h_i|<2^{-k_i},i\in e}\left\|\Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h)\Delta_h^{u,e} g_{k+\nu}(\cdot)|L_p(\mathbb{R}^d)\right\|\Bigg]^p\Bigg\}^{1/p} \\
& \le c_8\Big(\prod_{i=1}^{L} 2^{-|\nu_i|\delta}\Big)\Big(\prod_{i=L+1}^{N} 2^{-|\ell_i|\delta}\Big)\Bigg\{\sum_{k\in\mathbb{N}_0^d(e)}\ \sum_{\substack{\ell_i\in\mathbb{Z},i\notin\{L+1,\ldots,N\}\\ \nu_i\in\mathbb{Z},L+1\le i\le d}} 2^{|k+\ell|_1tp}\,2^{|k+\nu|_1tp} \\
& \qquad\qquad\qquad \times\quad \|f_{k+\ell}|L_p(\mathbb{R}^d)\|^p\,\|g_{k+\nu}|L_p(\mathbb{R}^d)\|^p\Bigg\}^{1/p},
\end{aligned}
\tag{3.11}
$$

see (3.10). Now we reorganize the summation in the curly bracket by putting $n:=k+\ell$ and $j=k+\nu$. Observe that for $\nu_1,\ldots,\nu_L$ and $\ell_{L+1},\ldots,L_N$ fixed the mapping

$$
(k_1,\ldots,k_N,\ell_1,\ldots,\ell_L,\ell_{N+1},\ldots,\ell_d,\nu_{L+1},\ldots,\nu_d)\mapsto(n_1,\ldots,n_d,j_1,\ldots,j_d)
$$

is one-to-one. Hence

$$
\begin{aligned}
& \sum_{k\in\mathbb{N}_0^d(e)}\ \sum_{\substack{\ell_i\in\mathbb{Z},i\notin\{L+1,\ldots,N\}\\ \nu_i\in\mathbb{Z},L+1\le i\le d}} 2^{|k+\ell|_1tp}2^{|k+\nu|_1tp}\|f_{k+\ell}|L_p(\mathbb{R}^d)\|^p\,\|g_{k+\nu}|L_p(\mathbb{R}^d)\|^p \\
& \qquad \le \sum_{n,j\in\mathbb{N}_0^d} 2^{|n|_1tp}2^{|j|_1tp}\|f_n|L_p(\mathbb{R}^d)\|^p\,\|g_j|L_p(\mathbb{R}^d)\|^p \\
& \qquad = \|\,f\,|S^t_{p,p}B(\mathbb{R}^d)\|^p\,\|\,g\,|S^t_{p,p}B(\mathbb{R}^d)\|^p.
\end{aligned}
\tag{3.12}
$$

Now we are in position to estimate $S_{e,u}$ under the given restrictions. From (3.11) and

(3.12) we derive

$$
\begin{aligned}
S_{e,u} &\le \Big\{ \sum_{k\in\mathbb{N}_0^d(e)} \Big[2^{t|k|_1} \sum_{\ell\in\mathbb{Z}^d} \sum_{\nu\in\mathbb{Z}^d} \sup_{|h_i|<2^{-k_i}, i\in e} \big\| \Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h) \Delta_h^{u,e} g_{k+\nu}(\cdot) | L_p(\mathbb{R}^d)\big\| \Big]^p \Big\}^{1/p} \\
&\le \sum_{\substack{\ell_i\in\mathbb{Z}, L<i\le N\\ \nu_i\in\mathbb{Z}, 1\le i\le L}} \Big\{ \sum_{k\in\mathbb{N}_0^d(e)} \Big[2^{t|k|_1} \sum_{\substack{\ell_i\in\mathbb{Z}, i\notin\{L+1,\ldots,N\}\\ \nu_i\in\mathbb{Z}, L+1\le i\le d}} \sup_{|h_i|<2^{-k_i}, i\in e} \big\| \ldots \big| L_p(\mathbb{R}^d)\big\| \Big]^p \Big\}^{1/p} \\
&\le c_8 \sum_{\substack{\ell_i\in\mathbb{Z}, L<i\le N\\ \nu_i\in\mathbb{Z}, 1\le i\le L}} \Big(\prod_{i=1}^{L} 2^{-|\nu_i|\delta} \Big) \Big(\prod_{i=L+1}^{N} 2^{-|\ell_i|\delta} \Big) \| f \,| S^t_{p,p}B(\mathbb{R}^d)\| \cdot \| g \,| S^t_{p,p}B(\mathbb{R}^d)\| \\
&\le c_9 \| f \,| S^t_{p,p}B(\mathbb{R}^d)\| \cdot \| g \,| S^t_{p,p}B(\mathbb{R}^d)\|
\end{aligned}
$$

with c_9 independent of f and g. This proves the claim in case $t > 1/p$.
Substep 3.2. Let $p = 1$ and $t = 1$. In this case we have

$$
S_{e,u} \le \sum_{k\in\mathbb{N}_0^d(e)} \sum_{\ell\in\mathbb{Z}^d} \sum_{\nu\in\mathbb{Z}^d} 2^{|k|_1} \sup_{|h_i|<2^{-k_i},\, i\in e} \big\| \Delta_h^{2\bar{m}-u,e} f_{k+\ell}(\cdot + u\diamond h) \Delta_h^{u,e} g_{k+\nu}(\cdot) \,\big| L_1(\mathbb{R}^d)\big\| .
$$

Now we use (3.10) with $\varepsilon = 0$ and continue as in the previous substep.
Part II - necessity. Let $t > 0$ and $1 \le p \le \infty$. Then the isotropic Besov space $B^t_{p,p}(\mathbb{R})$ is an algebra if and only if either $t > 1/p$ or $t = p = 1$, see [129, Theorem 2.6.2/1], [130, Theorem 2.8.3] or [100, Theorem 4.6.4/1]. Hence, if either $t = 1/p$ for some $1 < p < \infty$ or $0 < t < 1/p$, $1 \le p < \infty$, there exist two sequences $\{f_n\}_{n\in\mathbb{N}} \subset B^t_{p,p}(\mathbb{R})$ and $\{g_n\}_{n\in\mathbb{N}} \subset B^t_{p,p}(\mathbb{R})$ such that

$$
\|f_n \cdot g_n | B^t_{p,p}(\mathbb{R})\| \ge n \, \|f_n | B^t_{p,p}(\mathbb{R})\| \cdot \|g_n | B^t_{p,p}(\mathbb{R})\| , \qquad n \in \mathbb{N} .
$$

Let $\Psi \in C_0^\infty(\mathbb{R})$, $\Psi \not\equiv 0$. For $n \in \mathbb{N}$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we define the sequences

$$
F_n(x) := f_n(x_1)\cdot\Psi(x_2)\cdot\ldots\cdot\Psi(x_d) \quad \text{and} \quad G_n(x) = g_n(x_1)\cdot\Psi(x_2)\cdot\ldots\cdot\Psi(x_d) .
$$

The cross-norm property of $S^t_{p,p}B(\mathbb{R}^d)$ yields $\{F_n\}_{n=1}^\infty \subset S^t_{p,p}B(\mathbb{R}^d)$ and $\{G_n\}_{n=1}^\infty \subset S^t_{p,p}B(\mathbb{R}^d)$. Using the cross-norm property once again we find

$$
\begin{aligned}
\| F_n \cdot G_n \,| S^t_{p,p}B(\mathbb{R}^d)\| &= \| f_n \cdot g_n \,| B^t_{p,p}(\mathbb{R})\| \cdot \|\Psi^2 | B^t_{p,q}(\mathbb{R})\|^{d-1} \\
&\ge n \, \|f_n | B^t_{p,p}(\mathbb{R})\| \cdot \|g_n | B^t_{p,p}(\mathbb{R})\| \cdot \|\Psi^2 | B^t_{p,p}(\mathbb{R})\|^{d-1}
\end{aligned}
$$

and

$$
\|F_n | S^t_{p,p}B(\mathbb{R}^d)\| \cdot \|G_n | S^t_{p,p}B(\mathbb{R}^d)\| = \|f_n | B^t_{p,p}(\mathbb{R})\| \cdot \|g_n | B^t_{p,p}(\mathbb{R})\| \cdot \|\Psi | B^t_{p,q}(\mathbb{R})\|^{2(d-1)} .
$$

This obviously disproves that $S^t_{p,p}B(\mathbb{R}^d)$ is a multiplication algebra. ■

To characterize $M(S^t_{p,p}B(\mathbb{R}^d))$ we need the so-called localization property for the Besov spaces $S^t_{p,p}B(\mathbb{R}^d)$. For its proof we need another characterization by differences. This time we shall work with pure differences (not with associated moduli of smoothness).

Lemma 3.13. *Let $1 \le p \le \infty$ and $t > 0$. Let $m \in \mathbb{N}$ such that $m > t$. A function $f \in L_p(\mathbb{R}^d)$ belongs to $S^t_{p,p}B(\mathbb{R}^d)$ if and only if*

$$T_e := \Big\{ \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i\in e} |h_i|^{-tp}\Big) \big\|\Delta_h^{\bar{m},e} f(\cdot)\big|L_p(\mathbb{R}^d)\big\|^p \prod_{i\in e} \frac{\mathrm{d}h_i}{|h_i|}\Big\}^{1/p} < \infty\,,$$

for all $e \subset [d]$. It follows that

$$\|f|S^t_{p,p}B(\mathbb{R}^d)\| := \|f|L_p(\mathbb{R}^d)\| + \sum_{e\subset[d], e\neq\emptyset} T_e$$

is an equivalent norm on $S^t_{p,p}B(\mathbb{R}^d)$.

Remark 3.14. We give some comments on the proof of Lemma 3.13. A proof of a slightly modified statement (integration with respect to the components h_i is taken on $\mathbb{R}$, not on $[-1,1]$) can be found in [104, Section 2.3.4] and [137]. The reduction to the case considered in Lemma 3.13 can be done by standard arguments, we omit details.

Proposition 3.15. *Let $1 \le p \le \infty$ and $t > 0$. Let $\psi \in C_0^\infty(\mathbb{R}^d)$ satisfy (3.2). Then*

$$\|f|S^t_{p,p}B(\mathbb{R}^d)\| \asymp \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S^t_{p,p}B(\mathbb{R}^d)\|^p\Big)^{1/p}$$

holds for all $f \in S^t_{p,p}B(\mathbb{R}^d)$ (usual modification for $p = \infty$).

Proof. We prove for $1 \le p < \infty$. The proof for $p = \infty$ is a modification.
Step 1. We shall prove that

$$\|f|S^t_{p,p}B(\mathbb{R}^d)\| \lesssim \Big(\sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S^t_{p,p}B(\mathbb{R}^d)\|^p\Big)^{1/p} \tag{3.13}$$

holds for all $f \in S^t_{p,p}B(\mathbb{R}^d)$. Again we shall work with the characterization by differences. Let m be a natural number such that $t < m \le t+1$. Then, applying (3.2), the compactness of the support of ψ and $|h|_\infty \le 1$, we conclude

$$\begin{aligned}
\|f|S^t_{p,p}B(\mathbb{R}^d)\|^p &\lesssim \sum_{e\subset[d]} \sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 p} \sup_{|h_i|<2^{-k_i}, i\in e} \Big\| \sum_{\mu\in\mathbb{Z}^d} |\Delta_h^m (f\psi_\mu)(\cdot)| \Big| L_p(\mathbb{R}^d)\Big\|^p \\
&\lesssim \sum_{e\subset[d]} \sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 p} \sup_{|h_i|<2^{-k_i}, i\in e} \sum_{\mu\in\mathbb{Z}^d} \big\|\Delta_h^m (f\psi_\mu)(\cdot)\big|L_p(\mathbb{R}^d)\big\|^p \\
&\lesssim \sum_{\mu\in\mathbb{Z}^d} \|\psi_\mu f|S^t_{p,p}B(\mathbb{R}^d)\|^p.
\end{aligned}$$

This proves (3.13).
Step 2. We shall prove the reverse direction to (3.13). In some sense we will follow the same strategy as in proof of Theorem 3.10. Within this step we will use the characterization of $S^t_{p,p}B(\mathbb{R}^d)$ given in Lemma 3.13.
Substep 2.1. Some preparations. Let $t < m \le t+1$. Clearly, in case $e = \emptyset$ we have

$$\sum_{\mu\in\mathbb{Z}^d} \|f\psi_\mu|L_p(\mathbb{R}^d)\|^p = \|f|L_p(\mathbb{R}^d)\|^p \le \|f|S^t_{p,p}B(\mathbb{R}^d)\|^p.$$

For $e \subset [d]$, $e \neq \emptyset$ we use

$$\Delta_h^{2\bar{m},e}(f \cdot \psi_\mu)(x) = \sum_{u \in \mathbb{N}_0^d(e),\, |u|_\infty \le 2m} \binom{2\bar{m}}{u} \Delta_h^{2\bar{m}-u,e} f(x + u \diamond h) \Delta_h^{u,e} \psi_\mu(x)\,, \qquad x, h \in \mathbb{R}^d\,,$$

see (3.7). Recall $2\bar{m} - u := (2m - u_1, \ldots, 2m - u_d)$. It remains to estimate the terms

$$S_{e,u} := \bigg\{ \sum_{\mu \in \mathbb{Z}^d} \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i \in e} |h_i|^{-tp} \Big) \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \Delta_h^{u,e} \psi_\mu(\cdot) \big| L_p(\mathbb{R}^d) \big\|^p \prod_{i \in e} \frac{\mathrm{d}h_i}{|h_i|} \bigg\}^{1/p}.$$

This will be done by using the same splitting into various cases as done in the proof of Theorem 3.10.

Substep 2.2. The case $u_i < m$ for all $i \in e$. By assumption ψ has compact support and therefore $\operatorname{supp} \psi_\mu$ is contained in a cube $Q(\mu, c)$ with center in μ and sidelength $c > 0$. Because of $|h|_\infty \le 1$ we find

$$|\Delta_h^{u,e} \psi_\mu(x)| = 0 \qquad \text{if} \qquad |x - \mu|_\infty > R := c + 2m\,. \tag{3.14}$$

For simplicity we denote the cube $Q(\mu, R)$ with center in μ and sidelength R by Q_μ. Obviously it holds

$$|\Delta_h^{2\bar{m}-u,e} f(x + u \diamond h) \Delta_h^{u,e} \psi_\mu(x)| \lesssim \|\psi|C(\mathbb{R}^d)\|\, |\Delta_h^{2\bar{m}-u,e} f(x + u \diamond h)|\,. \tag{3.15}$$

Combining (3.15) and (3.14) we derive

$$\begin{aligned}
S_{e,u} &\lesssim \bigg\{ \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i \in e} |h_i|^{-tp} \Big) \sum_{\mu \in \mathbb{Z}^d} \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \big| L_p(Q_\mu) \big\|^p \prod_{i \in e} \frac{\mathrm{d}h_i}{|h_i|} \bigg\}^{1/p} \\
&\lesssim \bigg\{ \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i \in e} |h_i|^{-tp} \Big) \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \big| L_p(\mathbb{R}^d) \big\|^p \prod_{i \in e} \frac{\mathrm{d}h_i}{|h_i|} \bigg\}^{1/p} \\
&\lesssim \|f|S^t_{p,p}B(\mathbb{R}^d)\|\,.
\end{aligned}$$

Substep 2.3. The case $u_i \ge m$ for all $i \in e$. Let $0 < \varepsilon < m - t$. Directly from the definition of the spaces $S^{t+\varepsilon}_{\infty,\infty}B(\mathbb{R}^d)$ we derive the inequality

$$\Big(\prod_{i \in e} |h_i|^{-(t+\varepsilon)} \Big) |\Delta_h^{u,e} \psi_\mu(x)| \le \|\psi_\mu|S^{t+\varepsilon}_{\infty,\infty}B(\mathbb{R}^d)\| = \|\psi|S^{t+\varepsilon}_{\infty,\infty}B(\mathbb{R}^d)\|\,. \tag{3.16}$$

This inequality, combined with (3.14), results in

$$\begin{aligned}
S_{e,u} &\lesssim \bigg\{ \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i \in e} |h_i|^{\varepsilon p} \Big) \sum_{\mu \in \mathbb{Z}^d} \big\| \Delta_h^{2\bar{m}-u,e} f(\cdot + u \diamond h) \big| L_p(Q_\mu) \big\|^p \prod_{i \in e} \frac{\mathrm{d}h_i}{|h_i|} \bigg\}^{1/p} \\
&\lesssim \|f|L_p(\mathbb{R}^d)\| \bigg\{ \int\limits_{[-1,1]^{|e|}} \prod_{i \in e} \frac{\mathrm{d}h_i}{|h_i|^{1-\varepsilon p}} \bigg\}^{1/p} \\
&\lesssim \|f|S^t_{p,p}B(\mathbb{R}^d)\|\,.
\end{aligned} \tag{3.17}$$

Step 3. The remaining cases. Let $e_1 := \{i \in e : m \le u_i \le 2m\}$ and $e_2 := e \backslash e_1$. Obviously $e_1, e_2 \neq \emptyset$. As in (3.16) we conclude

$$|\Delta_h^{u,e}\psi_\mu(x)| \lesssim \sup_{x\in\mathbb{R}^d} |\Delta_h^{u,e_1}\psi_\mu(x)| \le \|\psi|S^{t+\varepsilon}_{\infty,\infty}B(\mathbb{R}^d)\| \prod_{i\in e_1} |h_i|^{t+\varepsilon}.$$

In a similar way as in (3.17) we obtain

$$\begin{aligned} S_{e,u} &\lesssim \Big\{ \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i\in e} |h_i|^{-t} \prod_{i\in e_1} |h_i|^{t+\varepsilon}\Big)^p \sum_{\mu\in\mathbb{Z}^d} \big\|\Delta_h^{2\bar{m}-u,e} f(\cdot + u\diamond h)\big|L_p(Q_\mu)\big\|^p \prod_{i\in e} \frac{\mathrm{d}h_i}{|h_i|}\Big\}^{1/p} \\ &\lesssim \Big\{ \int\limits_{[-1,1]^{|e|}} \Big(\prod_{i\in e_2} |h_i|^{-t} \prod_{i\in e_1} |h_i|^{\varepsilon}\Big)^p \big\|\Delta_h^{2\bar{m}-u,e} f(\cdot + u\diamond h)\big|L_p(\mathbb{R}^d)\big\|^p \prod_{i\in e} \frac{\mathrm{d}h_i}{|h_i|}\Big\}^{1/p}. \end{aligned}$$

Next we apply the elementary inequality

$$\big\|\Delta_h^{2\bar{m}-u,e} f(\cdot + u\diamond h)\big|L_p(\mathbb{R}^d)\big\| \lesssim \big\|\Delta_h^{m,e_2} f(\cdot)\big|L_p(\mathbb{R}^d)\big\|$$

since $2m - u_i \ge m$ if $i \in e_2$. Hence, we get

$$\begin{aligned} S_{e,u} &\lesssim \Big\{ \int\limits_{[-1,1]^{|e_2|}} \Big(\prod_{i\in e_2} |h_i|^{-tp}\Big) \big\|\Delta_h^{m,e_2} f(\cdot)\big|L_p(\mathbb{R}^d)\big\|^p \prod_{i\in e_2} \frac{\mathrm{d}h_i}{|h_i|}\Big\}^{1/p} \Big\{ \int\limits_{[-1,1]^{|e_1|}} \prod_{i\in e_1} \frac{\mathrm{d}h_i}{|h_i|^{1-\varepsilon p}}\Big\}^{1/p} \\ &\lesssim \|f|S^t_{p,p}B(\mathbb{R}^d)\|. \end{aligned}$$

as a consequence of Lemma 3.13. This finishes the proof. ■

We are now in position to formulate the spaces $M(S^t_{p,p}B(\mathbb{R}^d))$. For the definition of the space $S^t_{p,p}B(\mathbb{R}^d)_{\mathrm{unif}}$, see Definition 3.6. Note that as a consequence of Theorem 3.10 the space $S^t_{p,p}B(\mathbb{R}^d)_{\mathrm{unif}}$ is independent of the special choice of ψ in the sense of equivalent norms.

Theorem 3.16. *Let either $1 < p \le \infty$ and $t > 1/p$ or $p = 1$ and $t \ge 1$. Then*

$$M(S^t_{p,p}B(\mathbb{R}^d)) = S^t_{p,p}B(\mathbb{R}^d)_{\mathrm{unif}}$$

holds in the sense of equivalent norms.

Proof. By employing Proposition 3.15, Theorem 3.10 and similar arguments as in the proof of Theorem 3.8 one obtains the claimed identity $M(S^t_{p,p}B(\mathbb{R}^d)) = S^t_{p,p}B(\mathbb{R}^d)_{\mathrm{unif}}$. ■

Theorem 3.17. *Let $d > 1$, $1 \le p \le \infty$ and $t > 1/p$. Then there exists no constant $C > 0$ such that*

$$\|f\cdot g|S^t_{p,p}B(\mathbb{R}^d)\| \le C\left(\|f|S^t_{p,p}B(\mathbb{R}^d)\|\cdot\|g|L_\infty(\mathbb{R}^d)\| + \|f|L_\infty(\mathbb{R}^d)\|\cdot\|g|S^t_{p,p}B(\mathbb{R}^d)\|\right)$$

holds for all $f, g \in S^t_{p,p}B(\mathbb{R}^d)$.

Proof. Let $f \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp} f \subset [-2,2]$, $f(\xi) = 1$ if $\xi \in [-1,1]$. For $n \in \mathbb{N}$ we define $f_n(\xi) = f(2^n \xi)$, $\xi \in \mathbb{R}$. We recall that

$$\|h|B^t_{p,p}(\mathbb{R})\| \asymp \|h|L_p(\mathbb{R})\| + \|h|\dot{B}^t_{p,p}(\mathbb{R})\|$$

for all $h \in B^t_{p,p}(\mathbb{R})$ with $1 \le p \le \infty$ and $t > 0$ where $\dot{B}^t_{p,p}(\mathbb{R})$ are homogeneous Besov spaces, see [130, Section 5.2.3]. Hence we have

$$\|f_n|C(\mathbb{R})\| = 1 \qquad \text{and} \qquad \|f_n|B^t_{p,p}(\mathbb{R})\| \asymp 2^{n(t-1/p)}.$$

By defining $g \in C_0^\infty(\mathbb{R})$ with $g(\xi) = 1$ if $\xi \in [-2,2]$ we have

$$\|g|C(\mathbb{R})\| \asymp \|g|B^t_{p,p}(\mathbb{R})\| \asymp 1 \qquad \text{and} \qquad \|f_n g|B^t_{p,p}(\mathbb{R})\| \asymp 2^{n(t-1/p)}.$$

For $x \in \mathbb{R}^d$ we put

$$F_n(x) = f_n(x_1) \prod_{i=2}^d g(x_i) \qquad \text{and} \qquad G_n(x) = g(x_1) f_n(x_2) \prod_{i=3}^d g(x_i).$$

By cross-norm property, it follows that

$$\|F_n G_n|S^t_{p,p}B(\mathbb{R}^d)\| \asymp 2^{2n(t-1/p)}$$

and

$$\|F_n|S^t_{p,p}B(\mathbb{R}^d)\| \cdot \|G_n|C(\mathbb{R}^d)\| = \|G_n|S^t_{p,p}B(\mathbb{R}^d)\| \cdot \|F_n|C(\mathbb{R}^d)\| \asymp 2^{n(t-1/p)}.$$

This proves the assertion. ■

3.1.3 Pointwise multipliers for Sobolev-Besov spaces on the unit cube

Our main results obtained in the previous subsections carry over to the local case.

Theorem 3.18. (i) *Let $m \in \mathbb{N}$ and $1 < p < \infty$. Then $S^m_p W(\Omega)$ is a multiplication algebra.*
(ii) *Let $1 \le p \le \infty$ and $t > 0$. Then $S^t_{p,p}B(\Omega)$ is a multiplication algebra if either $1 < p \le \infty$ and $t > 1/p$ or $p = 1$ and $t \ge 1$.*

Proof. From Definition 1.58 we have for any $f \in S^m_p W(\Omega)$ there exists a function $f_{\text{ext}} \in S^m_p W(\mathbb{R}^d)$ such that

$$\|f|S^m_p W(\Omega)\| \le \|f_{\text{ext}}|S^m_p W(\mathbb{R}^d)\| \le 2\|f|S^m_p W(\Omega)\|\,.$$

Let $f, g \in S^m_p W(\Omega)$. Then from Theorem 3.1 we conclude that $f_{\text{ext}} g_{\text{ext}} \in S^m_p W(\mathbb{R}^d)$ and $f_{\text{ext}} g_{\text{ext}}$ is an extension of fg. We have

$$\begin{aligned}
\|fg|S^m_p W(\Omega)\| &\le \|f_{\text{ext}} g_{\text{ext}}|S^m_p W(\mathbb{R}^d)\| \\
&\le c_1\, \|f_{\text{ext}}|S^m_p W(\mathbb{R}^d)\| \cdot \|g_{\text{ext}}|S^m_p W(\mathbb{R}^d)\| \\
&\le 4c_1\, \|f|S^m_p W(\Omega)\| \cdot \|g|S^m_p W(\Omega)\|\,.
\end{aligned}$$

This proves the assertion (i). Proof of (ii) follows similarly. ■

Similarly as in the global case Theorem 3.18 can be turned into a characterizations of $M(S^m_pW(\Omega))$ and $M(S^t_{p,p}B(\Omega))$, respectively.

Theorem 3.19. (i) *Let* $m \in \mathbb{N}$ *and* $1 < p < \infty$. *Then*

$$M(S^m_pW(\Omega)) = S^m_pW(\Omega)$$

holds in the sense of equivalent norms.
(ii) *Let either* $1 < p \le \infty$ *and* $t > 1/p$ *or* $p = 1$ *and* $t \ge 1$. *Then*

$$M(S^t_{p,p}B(\Omega)) = S^t_{p,p}B(\Omega)$$

holds in the sense of equivalent norms.

Proof. The embedding $S^m_pW(\Omega) \hookrightarrow M(S^m_pW(\Omega))$ follows from the algebra property. If we assume $f \in M(S^m_pW(\Omega))$ we conclude that

$$\|f \cdot g|S^m_pW(\Omega)\| \le c\,\|f|M(S^m_pW(\Omega))\| \cdot \|g|S^m_pW(\Omega)\|$$

holds for all $g \in S^m_pW(\Omega)$. But the function $g \equiv 1$ (on Ω) belongs to $S^m_pW(\Omega)$. Hence, f must be an element of $S^m_pW(\Omega)$. Similarly we argue in case of $S^t_{p,p}B(\Omega)$. ■

Also in the local situation a Moser-type inequality does not hold.

Theorem 3.20. *Let* $d > 1$.
(i) *Let* $1 < p < \infty$ *and* $m \in \mathbb{N}$. *There exists no constant* $C > 0$ *such that*

$$\|f \cdot g|S^m_pW(\Omega)\| \le C\big(\|f|S^m_pW(\Omega)\| \cdot \|g|L_\infty(\Omega)\| + \|f|L_\infty(\Omega)\| \cdot \|g|S^m_pW(\Omega)\|\big)$$

holds for all $f, g \in S^m_pW(\Omega)$.
(ii) *Let* $1 \le p \le \infty$ *and* $t > 1/p$. *Then there exists no constant* $C > 0$ *such that*

$$\|f \cdot g|S^t_{p,p}B(\Omega)\| \le C\,\big(\|f|S^t_{p,p}B(\Omega)\| \cdot \|g|L_\infty(\Omega)\| + \|f|L_\infty(\Omega)\| \cdot \|g|S^t_{p,p}B(\Omega)\|\big)$$

holds for all $f, g \in S^t_{p,p}B(\Omega)$.

Proof. All test functions used in this context for the proof on $\mathbb{R}^d$ had compact support. From this remark, Theorem 3.20 follows. ■

Remark 3.21. Let us give a final comment for this section. By using a similar argument as in proof of Theorem 3.10 we can prove that the space $S^t_{p,q}B(\mathbb{R}^d)$ is a multiplication algebra if $t > 1/p$ and $0 < p, q \le \infty$. Concerning the Bessel-Potential spaces $S^t_pH(\mathbb{R}^d)$ we can extend the result in Theorem 3.1 to $1 < p < \infty$ and $t > \max(1/p, 1 - 1/p)$ by employing the multi-linear interpolation in the sense of Calderón [16], for more details see [68]. The case $2 < p < \infty$ and $1/p < t < 1 - 1/p$ remains open. A further interesting open problem is to characterize the space of all pointwise multipliers for $S^t_{p,q}B(\mathbb{R}^d)$ with $p \ne q$ and $t > 1/p$.

3.2 Change of variable operators and numerical integration of functions on the unit cube

3.2.1 Change of variable in spaces of dominating mixed smoothness

Let $\varphi \in C_0^r(\mathbb{R})$ such that $\operatorname{supp}\varphi \subset [0,1]$, $\int_0^1 \varphi(\xi)\,\mathrm{d}\xi = 1$, $\varphi(\xi) > 0$ on $(0,1)$ and the rth derivative $\varphi^{(r)}$ has only finitely many zeros in $[0,1]$. Let further

$$\psi(\xi) := \int_{-\infty}^{\xi} \varphi(s)\,\mathrm{d}s, \quad \xi \in \mathbb{R}. \tag{3.18}$$

Then $\psi'(\xi) = \varphi(\xi)$ for $\xi \in \mathbb{R}$. A natural and simple choice of ψ is given by the family of polynomials

$$\psi_r(\xi) = \Big(\int_0^1 s^r(1-s)^r\,\mathrm{d}s\Big)^{-1} \int_0^{\xi} s^r(1-s)^r\,\mathrm{d}s \tag{3.19}$$

if $\xi \in [0,1]$, $\psi_r(\xi) = 0$ if $\xi < 0$ and $\psi_r(\xi) = 1$ if $\xi > 1$. Another possible choice is the C^∞-kernel

$$\psi_\infty(\xi) = \Big(\int_0^1 e^{-\frac{1}{s(1-s)}}\,\mathrm{d}s\Big)^{-1} \int_0^{\xi} e^{-\frac{1}{s(1-s)}}\,\mathrm{d}s$$

if $\xi \in [0,1]$, $\psi_\infty(\xi) = 0$ if $\xi < 0$ and $\psi_\infty(\xi) = 1$ if $\xi > 1$. For $f \in L_1(\mathbb{R}^d)$ the "change of variable" operator is defined as

$$T_\Psi\colon\ f(x) \ \mapsto\ \Phi(x)f(\Psi(x)) := \Big(\prod_{i=1}^{d}\varphi(x_i)\Big) f\big(\psi(x_1),\ldots,\psi(x_d)\big), \quad x \in \mathbb{R}^d.$$

In this section we will study under which condition T_Ψ yields a bounded operator from the space $S^t_{p,q}A(\mathbb{R}^d)$ into itself.

It has been proved by Bykovskii [12] that T_{Ψ_r} is bounded on the Sobolev space with dominating mixed smoothness $S^t_2W(\mathbb{R}^d)$, $t \in \mathbb{N}$, if $r \geq 2t+1$. This result has been extended by Temlyakov, see [120, Theorem IV.4.1], to Sobolev spaces $S^t_pW(\mathbb{R}^d)$, $t \in \mathbb{N}$, $1 < p < \infty$, under the condition

$$r \geq \Big[\frac{tp}{p-1}\Big] + 1. \tag{3.20}$$

Under the same condition, Temlyakov [120, Lemma IV.4.9] showed the boundedness of T_{Ψ_r} in the Hölder-Nikol'skij spaces $S^t_{p,\infty}B(\mathbb{R}^d)$ if $1 < p \leq \infty$ and $t > 1$.

Concerning the C^∞-kernel the boundedness of the operator T_{Ψ_∞} has been studied by Dubinin [25, 26] for the Besov spaces $S^t_{p,q}B(\mathbb{R}^d)$ if $1 \leq p,q \leq \infty$, $t > 1/p$ and by Temlyakov [123] for the spaces $S^t_pW(\mathbb{R}^d)$, $t \in \mathbb{N}$, $1 \leq p \leq \infty$. Our results in this section read as follows.

Theorem 3.22. *Let $1 \leq p \leq \infty$, $0 < q \leq \infty$, $t > 0$ and $\varphi \in C_0^r(\mathbb{R})$ as above with $r > [t]+1$ if $p > 1$ and $r > [t]+2$ if $p = 1$. Then there exists a constant $C > 0$ such that*

$$\|T_\Psi f|S^t_{p,q}B(\mathbb{R}^d)\| \ \leq\ C\cdot\|f|S^t_{p,q}B(\mathbb{R}^d)\|$$

holds for all $f \in S^t_{p,q}B(\mathbb{R}^d)$.

Theorem 3.23. *Let $1 < p < \infty$, $1 < q \le \infty$, $t > 0$ and $\varphi \in C_0^r(\mathbb{R})$ as above with $r > [t]+1$. Then there exists a constant $C > 0$ such that*

$$\|T_\Psi f|\, S^t_{p,q}F(\mathbb{R}^d)\| \;\le\; C \cdot \|f|\, S^t_{p,q}F(\mathbb{R}^d)\|$$

holds for all $f \in S^t_{p,q}F(\mathbb{R}^d)$.

Remark 3.24. Observe that the smoothness r of the kernel φ in the F-case does not have to grow to infinity when p tends to 1. This result has to be compared with the already mentioned result of Temlyakov, see (3.20) and [120, page 237].

To prepare the proof of Theorems 3.22 and 3.23, let us first study the boundedness of quotients of derivatives of φ. For the particular choice $\psi = \psi_r$, see (3.19), the lemma below has been proved by Temlyakov [120, page 238].

Lemma 3.25. *Let $1 < p \le \infty$ and $m, r \in \mathbb{N}_0$ such that $r > \frac{mp}{p-1}+1$. Let further $\varphi \in C_0^r(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [0,1]$ and $\varphi > 0$ on $(0,1)$. Assume that the rth derivative $\varphi^{(r)}$ has only finitely many zeros in $[0,1]$. Then we have*

$$\frac{|\varphi^{(n)}(\xi)|}{\varphi(\xi)^{1/p}} \in L_\infty([0,1])$$

for all $0 \le n \le m$.

Proof. It is enough to prove the assertion for $n = m$. If $p = \infty$ the result is obvious. Hence, we assume that $1 < p < \infty$. We put $\ell = r - 1$. Using Taylor's theorem and the fact that $\varphi^{(i)}(0) = 0$ for all $i = 0, \ldots, r$, we obtain

$$\varphi(\xi) = \frac{1}{\ell!}\int_0^\xi \varphi^{(\ell+1)}(s)\,(\xi - s)^\ell \,\mathrm{d}s$$

and

$$\varphi^{(m)}(\xi) = \frac{1}{(\ell-m)!}\int_0^\xi \varphi^{(\ell+1)}(s)\,(\xi - s)^{\ell-m} \,\mathrm{d}s$$

for all $\xi \in [0,1]$. Since $\varphi^{(\ell+1)}$ has only finitely many zeros in $[0,1]$ there exists an $\varepsilon > 0$ such that $\varphi^{(\ell+1)}(\xi) > 0$ for $\xi \in (0,\varepsilon)$. This shows with $p' = p/(p-1)$ that

$$\begin{aligned}
\frac{|\varphi^{(m)}(\xi)|}{\varphi(\xi)^{1/p}} &\le \frac{\ell!}{(\ell-m)!}\,\frac{\xi^{1/p'}\Big(\int_0^\xi \big(\varphi^{(\ell+1)}(s)\big)^p\,(\xi-s)^{p(\ell-m)}\,\mathrm{d}s\Big)^{1/p}}{\Big(\int_0^\xi \varphi^{(\ell+1)}(s)\,(\xi-s)^\ell\,\mathrm{d}s\Big)^{1/p}} \\
&\lesssim_r \sup_{s\in(0,\varepsilon)}\Big(|\varphi^{(\ell+1)}(s)|^{p-1}\,|\xi-s|^{p(\ell-m)-\ell}\Big)^{1/p}\frac{\Big(\int_0^\xi \varphi^{(\ell+1)}(s)\,(\xi-s)^\ell\,\mathrm{d}s\Big)^{1/p}}{\Big(\int_0^\xi \varphi^{(\ell+1)}(s)\,(\xi-s)^\ell\,\mathrm{d}s\Big)^{1/p}} \\
&= \sup_{s\in(0,\varepsilon)}\Big(|\varphi^{(\ell+1)}(s)|^{p-1}\,|\xi-s|^{p(\ell-m)-\ell}\Big)^{1/p}.
\end{aligned}$$

Because of $p > 1$ and $\ell > \frac{pm}{p-1}$ we conclude that $\frac{|\varphi^{(m)}(\xi)|}{\varphi(\xi)^{1/p}} \le C$ for all $\xi \in (0,\varepsilon)$. The same arguments work also for $(1-\varepsilon_0, 1)$ with some $\varepsilon_0 > 0$. The quotient is uniformly bounded in $[\varepsilon, 1-\varepsilon_0]$ since $\varphi(\xi) \ge c > 0$ for $\xi \in [\varepsilon, 1-\varepsilon_0]$. The proof is complete. ■

It is easily seen that Lemma 3.25 is not true for $p = 1$. We immediately obtain the following corollary.

Corollary 3.26. *Let $m, r \in \mathbb{N}$ and $r > m$. Let further $\varphi \in C_0^r(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [0,1]$ and $\varphi > 0$ on $(0,1)$. Assume that the rth derivative $\varphi^{(r)}$ has only finitely many zeros in $[0,1]$. Then we have*

$$\frac{|\varphi^{(\beta)}(\xi)\varphi^{(\alpha)}(\xi)|}{\varphi(\xi)} \in L_\infty([0,1])$$

for all $\beta, \alpha \in \mathbb{N}_0$ with $\beta + \alpha \leq m - 1$.

Proof. If $\beta = 0$ or $\alpha = 0$, the statement is obvious since $\varphi \in C_0^r(\mathbb{R})$. Hence, we assume that $\beta \neq 0$ and $\alpha \neq 0$. We choose $p_1 = (\beta + \alpha)/\alpha$, $p_2 = (\beta + \alpha)/\beta$ and write

$$\frac{|\varphi^{(\beta)}(\xi)\varphi^{(\alpha)}(\xi)|}{\varphi(\xi)} \;=\; \frac{|\varphi^{(\beta)}(\xi)|}{\varphi(\xi)^{1/p_1}} \cdot \frac{|\varphi^{(\alpha)}(\xi)|}{\varphi(\xi)^{1/p_2}}\,. \tag{3.21}$$

From Lemma 3.25 we conclude that the term on the right-hand side of (3.21) is bounded on $[0,1]$ if

$$r > \frac{\beta p_1}{p_1 - 1} + 1 \qquad \text{and} \qquad r > \frac{\alpha p_2}{p_2 - 1} + 1$$

which implies $r > \beta + \alpha + 1$. But this is guaranteed by our assumptions $\beta + \alpha \leq m - 1$ and $r > m$. We finish the proof. ■

Lemma 3.27. *Let $m, \beta, \alpha, r \in \mathbb{N}_0$ such that $\beta \geq 1$, $\alpha + \beta = m$ and $r > m$. Let further $\varphi \in C_0^r(\mathbb{R})$ with $\operatorname{supp}\varphi \subset [0,1]$ and $\varphi > 0$ on $(0,1)$. Assume that the rth derivative $\varphi^{(r)}$ has only finitely many zeros in $[0,1]$ and ψ is given in (3.18). Then there exists a constant $C > 0$ such that*

$$\left|\varphi^{(\alpha)}(\xi)[g(\psi)]^{(\beta)}(\xi)\right| \;\leq\; C \sum_{1\leq\gamma\leq\beta} \left|g^{(\gamma)}(\psi(\xi))\right| \cdot \varphi(\xi)$$

holds for all $g \in C^m(\mathbb{R})$ and $\xi \in \mathbb{R}$.

Proof. It is sufficient to prove the assertion for $\xi \in [0,1]$. Using Faà di Bruno's formula for the chain rules of higher derivatives we have

$$\varphi^{(\alpha)}(\xi)[g(\psi)]^{(\beta)}(\xi) \;=\; \sum_{1\leq\gamma\leq\beta} g^{(\gamma)}(\psi(\xi)) \cdot \varphi^{(\alpha)}(\xi) \cdot Q_\gamma(\xi)\,,$$

where $Q_\gamma(\xi)$ has the form

$$Q_\gamma(\xi) = \sum_{k_1,\ldots,k_\beta} \frac{\beta!}{k_1!k_2!\cdots k_\beta!}\Big(\frac{\varphi(\xi)}{1!}\Big)^{k_1}\Big(\frac{\varphi'(\xi)}{2!}\Big)^{k_2}\cdots\Big(\frac{\varphi^{(\beta-1)}(\xi)}{\beta!}\Big)^{k_\beta}$$

with the sum is taken over all nonnegative integer $k_1, \ldots, k_\beta$ such that

$$k_1 + 2k_2 + \cdots + \beta k_\beta = \beta \qquad \text{and} \qquad k_1 + k_2 + \cdots + k_\beta = \gamma.$$

Since the highest order derivative of φ in Q_γ is $\beta - 1$ and $\varphi \in C_0^r(\mathbb{R})$ with $r > m$, Corollary 3.26 implies the existence of a positive constant C such that

$$|\varphi^{(\alpha)}(\xi) \cdot Q_\gamma(\xi)| < C\varphi(\xi),$$

for all γ, $1 \leq \gamma \leq \beta$, and all $\xi \in [0,1]$. This finishes the proof. ■

Proof of Theorem 3.23. *Step 1.* The tool of our proof will be the characterization by rectangular means of differences, see Theorem 1.54. We choose $m = [t] + 1$. Using a change of variables in the L_p-integral we obtain

$$\|T_\Psi f|L_p(\mathbb{R}^d)\| \le C\|f|L_p(\mathbb{R}^d)\| \le C\|f|S^t_{p,q}F(\mathbb{R}^d)\|$$

for all $f \in S^t_{p,q}F(\mathbb{R}^d)$. Here we have used $p > 1$ and $\varphi \in C_0^r(\mathbb{R})$. This inequality can be interpreted as the estimate for the term with $e = \emptyset$. We now consider the case $e \subset [d]$, $e \neq \emptyset$. Let $\{\varphi_k\}_{k\in\mathbb{N}_0^d}$ be the smooth dyadic decomposition of unity given in Remark 1.24. Using the decomposition

$$f = \sum_{\ell\in\mathbb{Z}^d} f_{k+\ell}, \qquad f_{k+\ell} = \mathcal{F}^{-1}[\varphi_{k+\ell}\mathcal{F}f],$$

see (3.8), we have the estimate

$$\begin{aligned} \Big\|\Big(\sum_{k\in\mathbb{N}_0^d(e)} & 2^{t|k|_1 q}\mathcal{R}^e_{\bar{m}}(T_\Psi f, 2^{-k},\cdot)^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| \\ & \le \sum_{\ell\in\mathbb{Z}^d}\Big\|\Big(\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\mathcal{R}^e_{\bar{m}}(T_\Psi f_{k+\ell}, 2^{-k},\cdot)^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|. \end{aligned} \tag{3.22}$$

Here recall $2^{-k} = (2^{-k_1},\ldots,2^{-k_d})$ and $\varphi_k \equiv 0$ if $\min_{i=1,\ldots,d} k_i < 0$. If $\ell \in \mathbb{Z}^d$ then $\omega(\ell)$ and $\varpi(\ell)$ have the same meaning as in (3.9). Note that $k \in \mathbb{N}_0^d(e)$ implies $\omega(\ell) \subset e$. For $k \in \mathbb{N}_0^d(e)$ we denote

$$P_k = \{x = (x_1,\ldots,x_d) \in \mathbb{R}^d :\ |x_i| \le 2^{-k_i}\}.$$

For simplicity we put

$$F_\ell := \Big\|\Big(\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\mathcal{R}^e_{\bar{m}}(T_\Psi f_{k+\ell}, 2^{-k},\cdot)^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\|.$$

Step 2. Estimate of $F(\ell)$.
Substep 2.1. We have

$$\begin{aligned} |\Delta_h^{\bar{m},e}(T_\Psi f_{k+\ell})(x)| &= |\Delta_h^{\bar{m},\varpi(\ell)\cap e} \circ \Delta_h^{\bar{m},\omega(\ell)}(T_\Psi f_{k+\ell})(x)| \\ &\lesssim \sum_u \big|\Delta_h^{\bar{m},\omega(\ell)}(T_\Psi f_{k+\ell})(x + u\diamond h)\big|, \end{aligned} \tag{3.23}$$

where the sum is taken over all $u = (u_1,\ldots,u_d) \in \mathbb{N}_0^d(\varpi(\ell)\cap e)$ such that $0 \le u_i \le m$ if $i \in \varpi(\ell)\cap e$. Note that if ϕ is a function with $\phi^{(k)}$, $k = 1,\ldots,m$, are locally integrable, then

$$\Delta_h^m\phi(t) = h^{m-1}\int_{\mathbb{R}} \phi^{(m)}(t+\xi)B_m(h^{-1}\xi)\,\mathrm{d}\xi\,,\ h > 0\,, \tag{3.24}$$

see [23, page 45]. Here $B_m(\cdot)$ is the univariate B-spline of degree m which has knots at the points $\{0, 1,\ldots,m\}$, i.e.,

$$B_m = \chi_{[0,1]} * \ldots * \chi_{[0,1]}, \qquad m \text{ times},$$

and $\chi_{[0,1]}$ denotes the characteristic function of $[0,1]$. Since $\operatorname{supp} B_m(h^{-1}\cdot) \subset [0,mh]$ and B_m is bounded, we have

$$\big|\Delta_h^m(\phi,t)\big| = \left|h^{m-1}\int\limits_0^{mh} \phi^{(m)}(t+\xi)B_m(h^{-1}\xi)\,\mathrm{d}\xi\right| \lesssim |h|^m\mathcal{M}(\phi^{(m)})(t).$$

Recall $\mathcal{M}$ here is the Hardy-Littlewood maximal operator, see (1.1). If $h<0$, by using the B-spline with knots at the points $\{-m,\ldots,-1,0\}$, we obtain a similar estimate. Applying the above inequality with the components in $\omega(\ell)$ we have found

$$\begin{aligned}&\Delta_h^{\bar{m},\omega(\ell)}(T_\Psi f_{k+\ell})(x+u\diamond h)\big|\\ &\qquad\lesssim \Big(\prod_{i\in\omega(\ell)} 2^{-k_i m}\Big)\Big(\prod_{i\in\omega(\ell)}\mathcal{M}_i\Big)\big[D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})\big](x+u\diamond h)\,.\end{aligned}$$

Here we use the notation $D^{(\bar{m},\omega(\ell))}=D^\beta$ with $\beta_i=m$ if $i\in\omega(\ell)$ and otherwise $\beta_i=0$. It follows that

$$\begin{aligned}\mathcal{R}^e_{\bar{m}}(T_\Psi f_{k+\ell},2^{-k},x) &\lesssim \sum_u 2^{|k|_1}\int\limits_{P_k}\big|\Delta_h^{\bar{m},\omega(\ell)}(T_\Psi f_{k+\ell})(x+u\diamond h)\big|\,\mathrm{d}h\\ &\lesssim \sum_u 2^{|k|_1}\int\limits_{P_k}\Big(\prod_{i\in\omega(\ell)} 2^{-k_i m}\Big)\Big(\prod_{i\in\omega(\ell)}\mathcal{M}_i\Big)\big[D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})\big](x+u\diamond h)\,\mathrm{d}h\,.\end{aligned}$$

By applying the Hardy-Littlewood maximal function with the components in $\overline{\omega}(\ell)\cap e$ we obtain

$$\mathcal{R}^e_{\bar{m}}(T_\Psi f_{k+\ell},2^{-k},x) \lesssim \Big(\prod_{i\in\omega(\ell)} 2^{-k_i m}\Big)\mathcal{M}_{[d]}\big[D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})\big](x)\,.$$

Plugging this into $F(\ell)$, Theorem 1.2 yields

$$\begin{aligned}F(\ell) &\lesssim \Bigg\|\Bigg\{\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\Big(\prod_{i\in\omega(\ell)} 2^{-k_i mq}\Big)\Big(\mathcal{M}_{[d]}\big[D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})\big](\cdot)\Big)^q\Bigg\}^{1/q}\Bigg|L_p(\mathbb{R}^d)\Bigg\|\\ &\lesssim \Bigg\|\Bigg\{\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\Big(\prod_{i\in\omega(\ell)} 2^{-k_i mq}\Big)\big|D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})(\cdot)\big|^q\Bigg\}^{1/q}\Bigg|L_p(\mathbb{R}^d)\Bigg\|\,.\end{aligned}$$

Observe that if $\omega(\ell)=\emptyset$, i.e., $\ell_i\geq 0$ for all $i=1,\ldots,d$, then the above inequality becomes

$$\begin{aligned}F(\ell) &\lesssim \Bigg\|\Bigg\{\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\big|(T_\Psi f_{k+\ell})(\cdot)\big|^q\Bigg\}^{1/q}\Bigg|L_p(\mathbb{R}^d)\Bigg\|\\ &= \Bigg\|\Bigg\{\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\Big|\Big(\prod_{i=1}^d\varphi(x_i)\Big)f_{k+\ell}\big(\psi(x_1),\ldots,\psi(x_d)\big)\Big|^q\Bigg\}^{1/q}\Bigg|L_p(\mathbb{R}^d)\Bigg\|\\ &\lesssim \Bigg\|\Bigg\{\sum_{k\in\mathbb{N}_0^d(e)} 2^{t|k|_1 q}\big|f_{k+\ell}(\cdot)\big|^q\Bigg\}^{1/q}\Bigg|L_p(\mathbb{R}^d)\Bigg\|\\ &= \Big(\prod_{i:\,\ell_i\geq 0} 2^{-t\ell_i}\Big)\|f|\,S^t_{p,q}F(\mathbb{R}^d)\|\,.\end{aligned}\tag{3.25}$$

There in the thirst step we changed variables in L_p-integral and used $\varphi \in C_0^r(\mathbb{R})$. In case $\omega(\ell) \neq \emptyset$, Leibniz's formula results in

$$\begin{aligned}\left|D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})(x)\right| &= \left|D^{(\bar{m},\omega(\ell))}\big(\Phi f_{k+\ell}(\Psi)\big)(x)\right| \\ &\lesssim \sum_{\alpha,\beta} \left|D^\alpha \Phi(x) D^\beta[f_{k+\ell}(\Psi)](x)\right|.\end{aligned}$$

Here the sum on the right-hand side is taken over all $\alpha, \beta \in \mathbb{N}_0^d(\omega(\ell))$ with $\alpha_i + \beta_i = m$ if $i \in \omega(\ell)$. This leads to

$$F(\ell) \lesssim \sum_{\alpha,\beta} \left\| \left\{ \sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 q} \Big(\prod_{i \in \omega(\ell)} 2^{-k_i m q} \Big) \left|D^\alpha \Phi(\cdot) D^\beta [f_{k+\ell}(\Psi)](\cdot)\right|^q \right\}^{1/q} \middle| L_p(\mathbb{R}^d) \right\|. \tag{3.26}$$

Substep 2.2. For each α we put $\omega_1(\ell) := \{i \in \omega(\ell) : \alpha_i < m\}$ and $\omega_2(\ell) := \omega(\ell) \backslash \omega_1(\ell)$. Temporarily we assume that $\omega_1(\ell), \omega_2(\ell) \neq \emptyset$. Hence $\beta \in \mathbb{N}_0^d(\omega_1(\ell))$ and $1 \leq \beta_i \leq m$ if $i \in \omega_1(\ell)$. We have

$$\begin{aligned} &D^\alpha \Phi(x) D^\beta [f_{k+\ell}(\Psi)](x) \\ &\quad = \Big(\prod_{i \in \overline{\omega}(\ell)} \varphi(x_i) \prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \prod_{i \in \omega_1(\ell)} \varphi^{(\alpha_i)}(x_i) \Big) D^\beta [f_{k+\ell}(\Psi)](x). \end{aligned} \tag{3.27}$$

Since $\alpha_i + \beta_i = m < r$ for $i \in \omega_1(\ell)$, Lemma 3.27 can be applied with the components in $\omega_1(\ell)$ to obtain

$$\left| \Big(\prod_{i \in \omega_1(\ell)} \varphi^{(\alpha_i)}(x_i) \Big) D^\beta [f_{k+\ell}(\Psi)](x) \right| \lesssim \sum_\gamma \Big(\prod_{i \in \omega_1(\ell)} \varphi(x_i) \Big) \left| [D^\gamma f_{k+\ell}](\Psi(x)) \right|,$$

where the sum is taken over all $\gamma \in \mathbb{N}_0^d(\omega_1(\ell))$ such that $1 \leq \gamma_i \leq \beta_i$, $i \in \omega_1(\ell)$. Inserting this into (3.27) we have found

$$\left|D^\alpha \Phi(x) D^\beta [f_{k+\ell}(\Psi)](x)\right| \lesssim \sum_\gamma \Big(\prod_{i \in \overline{\omega}(\ell) \cup \omega_1(\ell)} \varphi(x_i) \prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) \left| [D^\gamma f_{k+\ell}](\Psi(x)) \right|.$$

Now the inequality $\varphi(\xi) \lesssim \varphi(\xi)^{1/p}$ for all $\xi \in \mathbb{R}$ yields

$$\begin{aligned} &\left|D^\alpha \Phi(x) D^\beta [f_{k+\ell}(\Psi)](x)\right| \\ &\quad \lesssim \sum_\gamma \Big(\prod_{i \in \omega_1(\ell) \cup \overline{\omega}(\ell)} \varphi(x_i)^{1/p} \Big) \left| \Big(\prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) [D^\gamma f_{k+\ell}](\Psi(x)) \right|. \end{aligned} \tag{3.28}$$

Putting (3.28) into (3.26) and changing variable with components in $\omega_1(\ell) \cup \overline{\omega}(\ell)$ we derive

$$\begin{aligned} F(\ell) \lesssim \sum_{\alpha,\beta,\gamma} \Bigg\| \Bigg\{ &\sum_{k \in \mathbb{N}_0^d(e)} \Big(2^{t|k|_1} \prod_{i \in \omega(\ell)} 2^{-k_i m} \Big)^q \\ &\times \quad \left| \Big(\prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) [D^\gamma f_{k+\ell}](z) \right|^q \Bigg\}^{1/q} \Bigg| L_p(\mathbb{R}^d) \Bigg\|. \end{aligned} \tag{3.29}$$

Here $z = (z_1(x_1), \ldots, z_d(x_d))$ with $z_i = \psi(x_i)$ if $i \in \omega_2(\ell)$ otherwise $z_i = x_i$.
Substep 2.3. We denote

$$G(\omega_2(\ell)) = \big\{ x \in \mathbb{R}^d : x_i \in [0,1] \ \text{ if } \ i \in \omega_2(\ell) \ \text{ and } \ x_i \in \mathbb{R} \ \text{ if } i \in \omega_1(\ell) \cup \overline{\omega}(\ell) \,\big\}.$$

It is obvious that

$$\Big| \Big(\prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) (D^\gamma f_{k+\ell})(z(x)) \Big| = 0$$

if $x \notin G(\omega_2(\ell))$. In case $x \in G(\omega_2(\ell))$ we have

$$\begin{aligned}
\Big| \Big(\prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) [D^\gamma f_{k+\ell}](z(x)) \Big| &\lesssim \sup_{x_i \in [0,1], i \in \omega_2(\ell)} (D^\gamma f_{k+\ell})(x) \\
&\le \sup_{y \in G(\omega_2(\ell))} \frac{|D^\gamma f_{k+\ell}(y)|}{\prod_{i \notin \omega_2(\ell)} (1 + 2^{k_i + \ell_i} |x_i - y_i|)^a} \\
&= \sup_{y \in G(\omega_2(\ell))} \frac{|D^\gamma f_{k+\ell}(y)| \prod_{i \in \omega_2(\ell)} (1 + 2^{k_i + \ell_i} |x_i - y_i|)^a}{\prod_{i=1}^d (1 + 2^{k_i + \ell_i} |x_i - y_i|)^a}.
\end{aligned}$$

The condition $x_i, y_i \in [0,1]$ if $i \in \omega_2(\ell)$ leads to

$$\begin{aligned}
\Big| \Big(\prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) [D^\gamma f_{k+\ell}](z(x)) \Big| &\lesssim \Big(\prod_{i \in \omega_2(\ell)} 2^{(k_i + \ell_i) a} \Big) \sup_{y \in G(\omega_2(\ell))} \frac{|D^\gamma f_{k+\ell}(y)|}{\prod_{i=1}^d (1 + 2^{k_i + \ell_i} |x_i - y_i|)^a} \\
&\le \Big(\prod_{i \in \omega_2(\ell)} 2^{(k_i + \ell_i) a} \Big) \sup_{y \in \mathbb{R}^d} \frac{|D^\gamma f_{k+\ell}(y)|}{\prod_{i=1}^d (1 + 2^{k_i + \ell_i} |x_i - y_i|)^a}.
\end{aligned}$$

Lemma 1.7 implies

$$\begin{aligned}
&\Big| \Big(\prod_{i \in \omega_2(\ell)} \varphi^{(m)}(x_i) \Big) (D^\gamma f_{k+\ell})(z(x)) \Big| \\
&\qquad \lesssim \Big(\prod_{i \in \omega_2(\ell)} 2^{(k_i + \ell_i) a} \prod_{i \in \omega_1(\ell)} 2^{(k_i + \ell_i) \gamma_i} \Big) P_{2^{k+\ell}, a} f_{k+\ell}(x) \\
&\qquad \lesssim \Big(\prod_{i \in \omega(\ell)} 2^{(k_i + \ell_i) m} \Big) P_{2^{k+\ell}, a} f_{k+\ell}(x)
\end{aligned} \tag{3.30}$$

with $m \ge a$. Inserting this into (3.29) we find

$$\begin{aligned}
F(\ell) &\lesssim \sum_{\alpha, \beta, \gamma} \Big\| \Big\{ \sum_{k \in \mathbb{N}_0^d(e)} \Big(2^{t|k|_1} \prod_{i \in \omega(\ell)} 2^{\ell_i m} \Big)^q \big(P_{2^{k+\ell}, a} f_{k+\ell} \big)^q \Big\}^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \\
&\lesssim \Big\| \Big\{ \sum_{k \in \mathbb{N}_0^d(e)} \Big(\prod_{i \in \omega(\ell)} 2^{\ell_i (m-t)} \prod_{i \in \overline{\omega}(\ell)} 2^{-t \ell_i} \Big)^q \big(2^{t|k+\ell|_1} P_{2^{k+\ell}, a} f_{k+\ell} \big)^q \Big\}^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \\
&\lesssim \Big\| \Big\{ \sum_{k \in \mathbb{N}_0^d(e)} \Big(\prod_{i:\, \ell_i < 0} 2^{\ell_i (m-t)} \prod_{i:\, \ell_i \ge 0} 2^{-t \ell_i} \Big)^q \big(2^{t|k+\ell|_1} P_{2^{k+\ell}, a} f_{k+\ell} \big)^q \Big\}^{1/q} \Big| L_p(\mathbb{R}^d) \Big\|
\end{aligned}$$

with $a > \max\{1/p, 1/q\}$. Since $m \geq 1$ and $p, q > 1$ we can find $a > \max\{1/p, 1/q\}$ such that $m > a$. Now Theorem 1.10 yields

$$F(\ell) \lesssim \Big(\prod_{i:\, \ell_i < 0} 2^{\ell_i(m-t)} \prod_{i:\, \ell_i \geq 0} 2^{-t\ell_i} \Big) \|f|\, S^t_{p,q}F(\mathbb{R}^d)\| \,. \tag{3.31}$$

Looking back on the above argument (with obvious modification) we find that the estimate (3.31) still holds true in case $\omega_1(\ell) = \emptyset$ or $\omega_2(\ell) = \emptyset$.
Step 3. Inserting (3.31) and (3.25) into (3.22) we conclude that

$$\Big\| \Big(\sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 q} \mathcal{R}^e_{\bar{m}}(T_\Psi f, 2^{-k}, \cdot)^q \Big)^{1/q} \Big| L_p(\mathbb{R}^d) \Big\| \;\lesssim\; \|f|\, S^t_{p,q}F(\mathbb{R}^d)\| \,.$$

This finishes the proof. ■

Proof of Theorem 3.22. *Step 1.* We use the same notation as in the proof of Theorem 3.23. Clearly, in case $e = \emptyset$ we have

$$\|T_\Psi f | L_p(\mathbb{R}^d)\| \leq C\|f|L_p(\mathbb{R}^d)\| \leq C\|f|S^t_{p,q}B(\mathbb{R}^d)\|$$

for all $f \in S^t_{p,q}B(\mathbb{R}^d)$. If $e \subset [d]$, $e \neq \emptyset$ we make use of Theorem 1.53 and the decomposition (3.8). Analogously to (3.22) we have

$$\begin{aligned}
\Big(\sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 q} \omega^e_{\bar{m}}(T_\Psi f, 2^{-k})^q_p \Big)^{1/q} &\leq \Big(\sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 q} \Big(\sum_{\ell \in \mathbb{Z}^d} \omega^e_m(T_\Psi f_{k+\ell}, 2^{-k})_p \Big)^q \Big)^{1/q} \\
&\leq \Big(\sum_{\ell \in \mathbb{Z}^d} \sum_{k \in \mathbb{N}_0^d(e)} 2^{t|k|_1 q} \omega^e_m(T_\Psi f_{k+\ell}, 2^{-k})^q_p \Big)^{1/q} .
\end{aligned}$$

Here we assume $q < 1$. If $q \geq 1$ we use triangle inequality.
Step 2. The case $p > 1$. We put $m = [t] + 1$. The proof is similar to the F-case but less technical. Here we use the classical Hardy-Littlewood maximal inequality (the scalar version of Theorem 1.1) and the scalar version of Theorem 1.10. These inequalities do not depend on the parameter q. Hence the statement in the case of Besov spaces can be extended to all q, i.e., $0 < q \leq \infty$.
Step 3. The case $p = 1$. Using the convolution representation of the mth order differences in $\omega(\ell)$, see (3.24), we obtain from (3.23)

$$\begin{aligned}
\Delta^{\bar{m},e}_h(T_\Psi f_{k+\ell})(x) &= \sum_u \Delta^{\bar{m},\omega(\ell)}_h (T_\Psi f_{k+\ell})(x + u \diamond h) \\
&= \sum_u \int_{\mathbb{R}^{|\omega(\ell)|}} D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})(x + y + u \diamond h) \prod_{i \in \omega(\ell)} h_i^{m-1} B_m(h_i^{-1} y_i)\, \mathrm{d}y_i \,.
\end{aligned}$$

Here $y_i = 0$ if $i \notin \omega(\ell)$. Inserting this into $\|\Delta^{\bar{m},e}_h(T_\Psi f_{k+\ell})(\cdot)|L_1(\mathbb{R}^d)\|$ and then changing the order of integration with the fact that $\int_{\mathbb{R}} h^{-1} B_m(h^{-1}\xi)\, \mathrm{d}\xi = 1$ we obtain

$$\omega^e_{\bar{m}}(f_{k+\ell}, 2^{-k})_1 \lesssim \Big(\prod_{i \in \omega(\ell)} 2^{-k_i m} \Big) \big\| D^{(\bar{m},\omega(\ell))}(T_\Psi f_{k+\ell})(\cdot) \big| L_1(\mathbb{R}^d) \big\| \,.$$

The next step is carried out as for F-spaces. Note that in this case we choose $m = [t] + 2$, since there must exist a such that $1 < a \leq m$ for the inequality (3.30) to hold. The proof is complete. ■

Remark 3.28. The last step of the proof for $p = 1$ shows that, based on our method, we have to guarantee that $\max\{1, t\} < m$. That's why we need the more restrictive condition $r > m+1 = [t]+2$ in Theorem 3.22. Under the additional assumption $t \geq 1$ we can relax this condition to $r > [t] + 1$ as in the case $p > 1$.

Remark 3.29. The argument in the proof of Theorem 3.23 does not work for the space $S^t_{1,q}F(\mathbb{R}^d)$ since the proof relies on the Fefferman-Stein maximal inequality, see Theorem 1.2. In the case $0 < p < 1$ the boundedness of T_Ψ remains open in both scales of function spaces.

3.2.2 Numerical integration of functions on the unit cube

Integration of multivariate functions plays a crucial role in many fields of mathematics and its applications. In most of the cases the integral can never be done analytically since often the integrated function may be known only at certain points or does not have closed-form expression. Therefore, they must be solved numerically. Let D_d be a domain in $\mathbb{R}^d$ and $\mathfrak{F}_d$ be a class of functions on D_d which is continuously embedded in $C(D_d)$. A cubature formula approximating the multivariate integral $I(f) = \int_{D_d} f(x)\, \mathrm{d}x$ of a function f in the class $\mathfrak{F}_d$ is given by

$$Q_n(f) := \sum_{i=1}^{n} \lambda_i f(x^i)\,, \tag{3.32}$$

where $X_n := \{x^1, \dots, x^n\} \subset D_d$ denotes the set of given integration nodes and $(\lambda_1, \dots, \lambda_n)$ denotes the vector of integration weights. The optimal error with respect to the class $\mathfrak{F}_d$ is defined as

$$\mathrm{Int}_n(\mathfrak{F}_d) := \inf_{Q_n} e(Q_n, \mathfrak{F}_d)\,, \quad n \in \mathbb{N}\,,$$

where the infimum is taken over all cubature formulae of the form (3.32) and

$$e(Q_n, \mathfrak{F}_d) := \sup_{\|f|\mathfrak{F}_d\| \leq 1} |I(f) - Q_n(f)|\,, \quad n \in \mathbb{N}\,.$$

The topic of numerically integrating multivariate functions go back to the work of Korobov [53], Hlawka [50] and Bakhvalov [3] in the 1960s. In the past 50 years this field has attracted a lot of interest, see Temlyakov [116, 118, 119, 123], Dubinin [25, 26], Skriganov [108], Triebel [133], Hinrichs [48], Novak, Woźniakowski [81], Dick, Pillichshammer [24], Dũng, Ullrich [31], and Markhasin [67] to mention at least a few. For the most recent publications in this direction we refer to [49, 57, 136].

In this section, we shall employ the boundedness of change of variable operators in Section 3.2.1 to study the relation between the worst-case integration errors for Besov-Triebel-Lizorkin spaces with dominating mixed smoothness on the unit cube $S^t_{p,q}A(\Omega)$ and functions supported strictly inside Ω

$$\mathring{S}^t_{p,q}A(\Omega) = \big\{ f \in S^t_{p,q}A(\mathbb{R}^d) : \ \operatorname{supp} f \subset \Omega \big\}.$$

We recall the already known results for $\mathring{S}^t_{p,q}A(\Omega)$.

Theorem 3.30. (i) *Let* $1 \le p \le \infty$, $0 < q \le \infty$ *and* $t > \frac{1}{p}$. *Then*

$$\mathrm{Int}_n(\mathring{S}^t_{p,q}B(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(1-\frac{1}{q})_+}, \quad n \ge 2.$$

(ii) *Let* $1 < p < \infty$, $1 < q \le \infty$ *and* $t > \max(\frac{1}{p}, \frac{1}{q})$. *Then*

$$\mathrm{Int}_n(\mathring{S}^t_{p,q}F(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(1-\frac{1}{q})}, \quad n \ge 2.$$

Let us give a brief comment on the proof of Theorem 3.30. For details we refer to [120, 123, 136]. The idea of the lower bound is to use the modern tool of atomic decompositions to construct appropriate local fooling functions. To estimate from above one uses the Frolov cubature formulas. The Frolov cubature formulas were introduced and studied in [37, 38]. For further discussion of this cubature we refer to [120, Chapter 4] and [123, 135, 136].

We proceed with the optimal cubature of the non-zero boundary function spaces. Since $\mathring{S}^t_{p,q}A(\Omega)$ is a subspace of $S^t_{p,q}A(\Omega)$ the lower bound of $\mathrm{Int}_n(\mathring{S}^t_{p,q}A(\Omega))$ is also a lower estimate of optimal error for the class $S^t_{p,q}A(\Omega)$. Concerning the upper bound, Frolov's method does not seem to be suitable for functions with non-zero boundary condition because the proof relies heavily on the use of Poisson's summation formula, see [136]. A classical approach towards upper bounds has been proposed by Bykovskii [12] by first performing a change of variable to obtain

$$\int_{\Omega} f(x)\ \mathrm{d}x = \int_{\Omega} |\det \Psi'(x)| f(\Psi(x))\ \mathrm{d}x \tag{3.33}$$

for some one-to-one differentiable mapping $\Psi : \Omega \mapsto \Omega$, and use afterwards a cubature formula (3.32) for the right-hand integrand in (3.33). The main observation is the fact that this approach results in a modified cubature formula

$$Q^{\Psi}_n(f) := Q_n\big(|\det \Psi'| \cdot f \circ \Psi\big) = \sum_{i=1}^{n} \lambda_i\, |\det \Psi'(x^i)|\, f(\Psi(x^i))\,. \tag{3.34}$$

At this point we need the function on the right-hand side of (3.34) to belong to $S^t_{p,q}A(\Omega)$. Hence, the next step consists in proving the preservation of mixed smoothness under the change of variable, in other word we prove that the operator

$$T_{\Psi} : S^t_{p,q}A(\mathbb{R}^d) \to S^t_{p,q}A(\mathbb{R}^d)$$

is bounded for suitably chosen $\Psi(x)$. This operator has been studied in Section 3.2.1. As already mentioned above, a straight-forward choice of $\Psi(x)$ is given by $\Psi_r(x) = (\psi_r(x_1), \dots, \psi_r(x_d))$ where

$$\psi_r(\xi) = \Big(\int_0^1 s^r(1-s)^r\ \mathrm{d}s\Big)^{-1} \int_0^{\xi} s^r(1-s)^r\,\mathrm{d}s$$

if $\xi \in [0,1]$, $\psi_r(\xi) = 0$ if $\xi < 0$ and $\psi_r(\xi) = 1$ if $\xi > 1$. Observe that with this choice the integrated function on the right-hand side of (3.33) belongs to the class $\mathring{S}^t_{p,q}A(\Omega)$ provided that $r > [t] + 1$. The above argument proves the following.

Theorem 3.31. *Let $1 \le p \le \infty$, $0 < q \le \infty$ and $t > 0$. Let further $r > [t] + 1$ ($r > [t] + 2$ if $p = 1$) and $\Psi_r(x) = (\psi_r(x_1), \ldots, \psi_r(x_d))$ as above. Then, provided that $S^t_{p,q}B(\mathbb{R}^d) \subset C(\mathbb{R}^d)$, a corresponding modified cubature formula* (3.34) *on $S^t_{p,q}B(\Omega)$ does not perform asymptotically worse than* (3.32) *performs on $\mathring{S}^t_{p,q}B(\Omega)$, i.e.,*

$$e(Q_n^{\Psi_r}, S^t_{p,q}B(\Omega)) \lesssim e(Q_n, \mathring{S}^t_{p,q}B(\Omega)) \,, \quad n \in \mathbb{N} \,.$$

An analogous results for the Triebel-Lizorkin spaces of dominating mixed smoothness $S^t_{p,q}F(\Omega)$ reads as follows.

Theorem 3.32. *Let $1 < p < \infty$, $1 < q \le \infty$ and $t > 0$. Let further $r > [t]+1$ and $\Psi(x) = (\psi_r(x_1), \ldots, \psi_r(x_d))$ as above. Then, provided that $S^t_{p,q}F(\mathbb{R}^d) \subset C(\mathbb{R}^d)$, a corresponding modified cubature formula* (3.34) *on $S^t_{p,q}F(\Omega)$ does not perform asymptotically worse than* (3.32) *performs on $\mathring{S}^t_{p,q}F(\Omega)$, i.e.,*

$$e(Q_n^{\Psi_r}, S^t_{p,q}F(\Omega)) \lesssim e(Q_n, \mathring{S}^t_{p,q}F(\Omega)) \,, \quad n \in \mathbb{N} \,.$$

These results are useful in the sense that they "transfer" the optimal rate of the minimal worst-case error from the known situation $\mathring{S}^t_{p,q}A(\Omega)$ to the more difficult situation $S^t_{p,q}A(\Omega)$. In view of Theorem 3.30 we obtain the asymptotic behaviour of $\mathrm{Int}_n(S^t_{p,q}A(\Omega))$.

Corollary 3.33. (i) *Let $1 \le p \le \infty$, $0 < q \le \infty$ and $t > \frac{1}{p}$. Then*

$$\mathrm{Int}_n(\mathring{S}^t_{p,q}B(\Omega)) \asymp \mathrm{Int}_n(S^t_{p,q}B(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(1-\frac{1}{q})_+}, \quad n \ge 2.$$

(ii) *Let $1 < p < \infty$, $1 < q \le \infty$ and $t > \max(\frac{1}{p}, \frac{1}{q})$. Then*

$$\mathrm{Int}_n(\mathring{S}^t_{p,q}F(\Omega)) \asymp \mathrm{Int}_n(S^t_{p,q}F(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(1-\frac{1}{q})}, \quad n \ge 2.$$

Remark 3.34. Let $S^t_{p,q}A(\mathbb{T}^d)$ be periodic Besov and Triebel-Lizorkin spaces of dominating mixed smoothness on the torus $\mathbb{T}^d = [0,1]^d$ and $\psi \in C^m_{\mathrm{mix}}(\mathbb{R}^d)$ with compact support and $m > t$. By employing the boundedness of the pointwise multiplication operator $f \to \psi f$ from $S^t_{p,q}A(\mathbb{T}^d)$ into $S^t_{p,q}A(\mathbb{R}^d)$ we can also prove that corresponding modified cubature formula of (3.32) on $S^t_{p,q}A(\mathbb{T}^d)$ does not perform asymptotically worse than (3.32) performs on $\mathring{S}^t_{p,q}A(\Omega)$ and as a consequence we obtain

$$\mathrm{Int}_n(\mathring{S}^t_{p,q}A(\Omega)) \asymp \mathrm{Int}_n(S^t_{p,q}A(\mathbb{T}^d)) \,, \quad n \in \mathbb{N} \,.$$

For further details we refer to [76], and also [136].

4 Weyl and Bernstein numbers of embeddings of tensor product Sobolev and Besov spaces

Let $\Omega := [0,1]^d$. In this chapter we shall investigate the asymptotic behaviour of Weyl and Bernstein numbers of embeddings of tensor product Besov and Sobolev spaces into Lebesgue spaces. As a matter of fact, tensor product Sobolev and Besov spaces are special cases of Sobolev and Besov spaces of dominating mixed smoothness, see Theorem 1.59. Hence, from now on we shall work with Weyl and Bernstein numbers of the identities

$$id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega) \qquad \text{and} \qquad id : S^t_{p_0}H(\Omega) \to L_p(\Omega).$$

We will write ω_n when we refer to both Weyl and Bernstein numbers. Weyl numbers of the embeddings $id : A^t_{p_0,q}(\Omega) \to L_p(\Omega)$ have been studied by Pietsch [86], Lubitz [65], König [55, Section 3.c] and Caetano [13, 14, 15]. Zhang, Fang, Huang [145] and Gasiorowska, Skrzypczak [43] investigated Weyl numbers of embeddings of weighted Besov spaces, defined on $\mathbb{R}^d$, into Lebesgue spaces. Bernstein numbers of embeddings of one-dimensional periodic Sobolev spaces were obtained by Tsarkov and Maiorov [125, page 194]. We refer to [58] and [75] for the case of higher dimensions. Galeev [40] studied Bernstein numbers of embeddings of periodic Sobolev spaces $S^t_pH(\mathbb{T}^d)$ and Nilol'skij spaces $S^t_{p,\infty}B(\mathbb{T}^d)$. We shall compare our results with those of Galeev in Section 4.5.3.

4.1 Weyl and Bernstein numbers

Let us first recall the notions of Weyl and Bernstein numbers.

Definition 4.1. Let X, Y be quasi-Banach spaces and $T \in \mathcal{L}(X,Y)$. Let $n \in \mathbb{N}$. Then the nth Weyl number of $T \in \mathcal{L}(X,Y)$ is defined as

$$x_n(T) := \sup\{a_n(TA) : \ A \in \mathcal{L}(\ell_2, X), \ \|A\| \le 1\}\,.$$

Here a_n is the nth approximation number which is given by

$$a_n(T) := \inf\{\|T - A : X \to Y\| : \ A \in \mathcal{L}(X,Y), \ \ \mathrm{rank}(A) < n\}\,, \qquad n \in \mathbb{N}\,.$$

Definition 4.2. Let X, Y be quasi-Banach spaces and $T \in \mathcal{L}(X,Y)$. Let $n \in \mathbb{N}$. Then the nth Bernstein number of $T \in \mathcal{L}(X,Y)$ is defined as

$$b_n(T) = \sup_{L_n} \inf_{\substack{x \in L_n \\ x \neq 0}} \frac{\|Tx \,|\, Y\|}{\|x \,|\, X\|},$$

where the supremum is taken over all subspaces L_n of X with dimension n.

We now recall the definition of s-numbers. We shall use the original notion by Pietsch [83] and [84, Chapter 11], see Remark 4.4. Although Pietsch has defined s-numbers for Banach spaces but we can extend this to the situation of quasi-Banach spaces. Let X, Y, X_0, Y_0 be quasi-Banach spaces. Let further Y be a ρ-Banach space for some $\rho \in (0,1]$, i.e.,

$$\|\, x + y \,|Y\|^\rho \le \|\, x \,|Y\|^\rho + \|\, y \,|Y\|^\rho \qquad \text{for all} \quad x,\, y \in Y\,. \tag{4.1}$$

The quasi-norm which satisfies (4.1) is called ρ-norm. It is clear that every Banach space is a 1-Banach space. Moreover, for every quasi-Banach space X, there exists a ρ-norm on X equivalent to the original norm. An s-function is a map s assigning to every operator $T \in \mathcal{L}(X,Y)$ a scalar sequence $\{s_n(T)\}_{n\in\mathbb{N}}$ such that the following conditions are satisfied:

(a) $\|T\| = s_1(T) \geq s_2(T) \geq \ldots \geq 0$;

(b) $s_n^\rho(S+T) \leq \|S\|^\rho + s_n^\rho(T)$ for all $S \in \mathcal{L}(X,Y)$ and $n \in \mathbb{N}$;

(c) $s_n(BTA) \leq \|B\| \cdot s_n(T) \cdot \|A\|$ for all $A \in \mathcal{L}(X_0, X)$, $B \in \mathcal{L}(Y, Y_0)$;

(d) $s_n(T) = 0$ if $\mathrm{rank}(T) < n$ for all $n \in \mathbb{N}$;

(e) $s_n(id : \ell_2^n \to \ell_2^n) = 1$ for all $n \in \mathbb{N}$.

Sometimes further properties are of some use. Let Z be a quasi-Banach space. An s-function is called additive if

(b') $s_{n+m-1}^\rho(S+T) \leq s_n^\rho(S) + s_m^\rho(T)$ for all $S, T \in \mathcal{L}(X,Y)$ and $m, n \in \mathbb{N}$;

and multiplicative if

(c') $s_{n+m-1}(ST) \leq s_n(S)\, s_m(T)$ for $T \in \mathcal{L}(X,Y)$, $S \in \mathcal{L}(Y,Z)$ and $m, n \in \mathbb{N}$.

According to the above definition, Weyl, Bernstein and approximation numbers are s-numbers. Concerning Weyl numbers, we have the following theorem, see Pietsch [85].

Theorem 4.3. *Weyl numbers are additive and multiplicative s-numbers.*

Remark 4.4. We observe that the conditions (b') and (c') contain (b) and (c) as a special case. In the literature there is some ambiguity concerning the notion of s-numbers. Pietsch, in [87, Chapter 2], has used a different definition of s-numbers in which he replaced axiom (b) by (b'), but later, in the monograph [88, Section 6.2], see also [89], he returned to his original definition. We wish to emphasize that according to the definition in [87, Chapter 2], Bernstein numbers are not s-numbers since they fail to be additive, see [89]. In the same paper, Pietsch also showed that Bernstein numbers are not multiplicative s-numbers.

There are some other s-numbers which are closed related to Weyl and Bernstein numbers. The nth Kolmogorov numbers of $T \in \mathcal{L}(X,Y)$ is defined as

$$d_n(T) = \inf_{L_{n-1}} \sup_{\|x|X\| \leq 1} \inf_{y \in L_{n-1}} \|Tx - y|Y\|. \tag{4.2}$$

Here the outer infimum is taken over all linear subspaces in X of dimension at most $n-1$. The nth Gelfand number of $T \in \mathcal{L}(X,Y)$ is given by

$$c_n(T) = \inf_{M_n} \sup_{\|x|X\| \leq 1, x \in M_n} \|Tx|Y\| \tag{4.3}$$

where M_n is a subspace of X such that $\mathrm{codim}(M_n) < n$. Approximation, Kolmogorov and Gelfand numbers are additive and multiplicative s-numbers, see, e.g., [84, Theorems 11.8.2, 11.9.2]. We collect some relations between Bernstein, Weyl numbers and other s-numbers.

Theorem 4.5. *Let X and Y be two quasi-Banach spaces and $T \in \mathcal{L}(X,Y)$. Then we have*

(i) $b_n(T) \leq c_n(T), d_n(T) \leq a_n(T)$,

(ii) $x_n(T) \le c_n(T) \le a_n(T)$,

(iii) $x_n(T) = c_n(T) = a_n(T)$ *if X is a Hilbert space,*

(iv) $b_n(T) = x_n(T) = c_n(T) = d_n(T) = a_n(T)$ *if X and Y are Hilbert spaces.*

Remark 4.6. We refer to [83] for the proof of part (i), [87, Chapter 2] for parts (ii), (iii) and [84, Theorem 11.3.4] for part (iv). Theorem 4.5 (iii) give us an alternative way to calculate the nth Weyl number. Indeed, for $T \in \mathcal{L}(X, Y)$ it holds

$$x_n(T) := \sup \Big\{ c_n(TA) : \; A \in \mathcal{L}(\ell_2, X), \; \|A\| \le 1 \Big\}, \tag{4.4}$$

see also Pietsch [85]. Sometime it is helpful to use this notation, e.g., in Theorem 4.11.

Lemma 4.7. *Let X, Y be Banach spaces and* $\dim(X) = \dim(Y) = m$. *If $T \in \mathcal{L}(X, Y)$ is invertible then*

$$b_n(T) c_{m-n+1}(T^{-1}) = 1 \, , \quad n \in \mathbb{N}, \; n \le m.$$

For a proof of Lemma 4.7 we refer to [83]. The relation between Weyl and Bernstein numbers reads as follows, see Pietsch [89].

Lemma 4.8. *Let X, Y be two Banach spaces and $T \in \mathcal{L}(X, Y)$. Then*

$$b_{2n-1}(T) \le e \Big(\prod_{k=1}^{n} x_k(T) \Big)^{1/n}, \quad n \in \mathbb{N}.$$

Corollary 4.9. *Let X, Y be two Banach spaces, $T \in \mathcal{L}(X, Y)$ and $\alpha > 0$, $\beta \ge 0$. Assume that $x_n(T) \asymp n^{-\alpha} (\log n)^\beta$, $n \ge 2$. Then*

$$b_n(T) \lesssim x_n(T),$$

for all $n \in \mathbb{N}$. Moreover, if Y is a Hilbert space we have $b_n(T) \asymp x_n(T)$.

Proof. The inequality of Pietsch and our assumption $x_n(T) \asymp n^{-\alpha} (\log n)^\beta$ yield

$$b_{2n-1}(T) \lesssim n^{-\alpha} (\log n)^\beta.$$

Now, if Y is a Hilbert space and $A \in \mathcal{L}(\ell_2, X)$ with $\|A\| \le 1$, from property (c) of the s-numbers and Theorem 4.5 (iv), we obtain

$$a_n(TA) = b_n(TA) \le b_n(T) \|A\| \le b_n(T).$$

The definition of Weyl numbers results in

$$x_n(T) \le b_n(T).$$

The proof is complete. ■

Next we consider the relation between Bernstein and entropy numbers. The nth (dyadic) entropy number of $T \in \mathcal{L}(X, Y)$ is defined as

$$e_n(T) := \inf \{ \varepsilon > 0 : T(B_X) \text{ can be covered by } 2^{n-1} \text{ balls in } Y \text{ of radius } \varepsilon \} \, ,$$

where $B_X := \{ x \in X : \; \|x|X\| \le 1 \}$ denotes the closed unit ball of X. Note that entropy numbers do not belong to the class of s-numbers since they do not satisfy the axiom (d). We have the following lemma. The proof is similar to the proof of Lemma 2 in [89].

Lemma 4.10. *Let X, Y be Banach spaces and $T \in \mathcal{L}(X, Y)$. Then we have*

$$b_n(T) \;\le\; 2\sqrt{2}\, e_n(T), \quad n \ge 1.$$

Proof. Without loss of generality we assume that $b_n(T) > 0$. Then for every $\varepsilon > 0$, $\varepsilon < b_n(T)$, there exists a linear subspace L_n of dimension n in X such that

$$0 < b_n(T) - \varepsilon \le \frac{\|Tx\,|\,Y\|}{\|x\,|\,X\|}, \quad \forall x \in L_n.$$

Denote by E the canonical embedding of L_n into X. Then TE induces an isomorphism S between L_n and $F_n := TE(L_n)$. It is obvious that $\|S^{-1} : F_n \to L_n\| \le (b_n(T) - \varepsilon)^{-1}$. By J we denote the canonical embedding from F_n into Y. Let us consider the diagram

$$\begin{tikzcd} X \arrow[r, "T"] & Y \\ L_n \arrow[u, "E"] \arrow[r, "S"] & F_n. \arrow[u, "J"'] \end{tikzcd}$$

By $\lambda_n(id : L_n \to L_n)$ we denote the nth eigenvalue of the identity $id : L_n \to L_n$. The Carl-Triebel inequality, see [19], and abstract property of entropy numbers yield

$$1 = \lambda_n(id : L_n \to L_n) \;\le\; \sqrt{2}\, e_n(id : L_n \to L_n) \;\le\; \sqrt{2}\, \|S^{-1}\|\, e_n(S).$$

Because J is an injection we have $e_n(S) \le 2e_n(JS)$, see [18, page 14]. Consequently we obtain

$$1 \;\le\; 2\sqrt{2}\, \|S^{-1}\|\, e_n(JS) \;=\; 2\sqrt{2}\, \|S^{-1}\|\, e_n(TE) \;\le\; 2\sqrt{2}\, \|S^{-1}\|\, e_n(T).$$

This implies

$$b_n(T) - \varepsilon \;\le\; 2\sqrt{2}\, e_n(T).$$

Letting $\varepsilon \downarrow 0$ we finish the proof. ■

Now we shall discuss the interpolation property of Weyl numbers. Interpolation properties of Kolmogorov and Gelfand numbers have been studied by Triebel [127] in the situation of Banach spaces. Without difficulty we can extend those result to the situation of quasi-Banach spaces. The following theorem shows that Gelfand and Weyl numbers share similar interpolation properties. Here we make use of formula (4.4).

Theorem 4.11. *Let $0 < \Theta < 1$. Let X, Y, X_0, Y_0 be quasi-Banach spaces. Further we assume $Y_0 \cap Y_1 \hookrightarrow Y$ and the existence of a positive constant C with*

$$\|y|Y\| \le C\, \|y|Y_0\|^{1-\Theta}\, \|y|Y_1\|^{\Theta} \qquad \text{for all} \quad y \in Y_0 \cap Y_1. \tag{4.5}$$

Then, if

$$T \in \mathcal{L}(X, Y_0) \cap \mathcal{L}(X, Y_1) \cap \mathcal{L}(X, Y)$$

it follows

$$c_{n+m-1}(T:\ X \to Y) \le C\, c_n^{1-\Theta}(T:\ X \to Y_0)\, c_m^{\Theta}(T:\ X \to Y_1)$$

and

$$x_{n+m-1}(T:\ X \to Y) \le C\, x_n^{1-\Theta}(T:\ X \to Y_0)\, x_m^{\Theta}(T:\ X \to Y_1)$$

for all $n, m \in \mathbb{N}$. Here C is the same constant as in (4.5).

Proof. *Step 1.* We follow the proof in [127] for Banach spaces. Let L_n and L_m be subspaces of X such that $\operatorname{codim}(L_n) < n$ and $\operatorname{codim}(L_m) < m$ respectively. Then $\operatorname{codim}(L_n \cap L_m) < m + n - 1$. Furthermore, by assumption, for all $x \in X$ we have $Tx \in Y_0 \cap Y_1$. From (4.3) and (4.5) we derive

$$\begin{aligned} c_{m+n-1}(T : X \to Y) &= \inf_{L_n, L_m} \sup_{\substack{\|x|X\| \le 1 \\ x \in L_n \cap L_m}} \|Tx|Y\| \\ &\le C \inf_{L_n, L_m} \sup_{\substack{\|x|X\| \le 1 \\ x \in L_n \cap L_m}} \| Tx \,|Y_0\|^{1-\Theta} \, \| Tx \,|Y_1\|^{\Theta} \\ &\le C \Big(\inf_{L_n} \sup_{\substack{\|x|X\| \le 1 \\ x \in L_n}} \| Tx \,|Y_0\| \Big)^{1-\Theta} \Big(\inf_{L_m} \sup_{\substack{\|x|X\| \le 1 \\ x \in L_m}} \| Tx \,|Y_1\| \Big)^{\Theta} \\ &= C \, c_n^{1-\Theta}(T : X \to Y_0) \, c_m^{\Theta}(T : X \to Y_1) \,. \end{aligned}$$

Step 2. Let $A \in \mathcal{L}(\ell_2, X)$ such that $\|A\| \le 1$. Then from Step 1 we conclude

$$c_{n+m-1}(TA : \ell_2 \to Y) \le C \, c_n^{1-\Theta}(TA : \ell_2 \to Y_0) \, c_m^{\Theta}(TA : \ell_2 \to Y_1).$$

Employing (4.4) we obtain

$$c_{n+m-1}(TA : \ell_2 \to Y) \le C \, x_n^{1-\Theta}(T : X \to Y_0) \, x_m^{\Theta}(T : X \to Y_1).$$

Now taking the supremum with respect to A we find

$$x_{n+m-1}(T : X \to Y) \le C \, x_n^{1-\Theta}(T : X \to Y_0) \, x_m^{\Theta}(T : X \to Y_1).$$

The proof is complete. ■

Remark 4.12. Triebel [127] worked with Gelfand widths. If T is a compact operator then the $(n+1)$th Gelfand number of the operator $T \in \mathcal{L}(X, Y)$ and the Gelfand n-width of $T(B_X)$ in Y coincide, see [127]. Here B_X denotes the closed unit ball of X. Without extra conditions on T Gelfand widths and Gelfand numbers may not coincide, see Edmunds and Lang [32] for a discussion of this question.

The interpolation property of Bernstein numbers in connection with Gelfand numbers reads as follows.

Theorem 4.13. *Let $0 < \Theta < 1$. Let X, Y, X_0, Y_0 be quasi-Banach spaces. Further we assume $Y_0 \cap Y_1 \hookrightarrow Y$ and the existence of a positive constant C with*

$$\|y|Y\| \le C \, \|y|Y_0\|^{1-\Theta} \, \|y|Y_1\|^{\Theta} \qquad \text{for all} \quad y \in Y_0 \cap Y_1. \tag{4.6}$$

Then, if

$$T \in \mathcal{L}(X, Y_0) \cap \mathcal{L}(X, Y_1) \cap \mathcal{L}(X, Y)$$

it follows

$$b_{n+m-1}(T : X \to Y) \le C \, c_n^{1-\Theta}(T : X \to Y_0) \, b_m^{\Theta}(T : X \to Y_1)$$

for all $n, m \in \mathbb{N}$. Here C is the same constant as in (4.6).

Proof. Let L_{m+n-1} be a linear subspace in X with $\dim(L_{m+n-1}) \geq m+n-1$. Let M_n be an arbitrary linear subspace in X such that $\operatorname{codim}(M_n) < n$. We put $L_m = L_{m+n-1} \cap M_n$. Then $\dim(L_m) \geq m$. We have

$$\begin{aligned}\inf_{\substack{x\in L_{m+n-1}\\ x\neq 0}} \frac{\|Tx|Y\|}{\|x|X\|} \leq \inf_{\substack{x\in L_m\\ x\neq 0}} \frac{\|Tx|Y\|}{\|x|X\|} &\leq C \inf_{\substack{x\in L_m\\ x\neq 0}} \frac{\|Tx|Y_0\|^{1-\Theta}}{\|x|X\|^{1-\Theta}} \cdot \frac{\|Tx|Y_1\|^{\Theta}}{\|x|X\|^{\Theta}} \\ &\leq C \sup_{\substack{x\in M_n\\ x\neq 0}} \frac{\|Tx|Y_0\|^{1-\Theta}}{\|x|X\|^{1-\Theta}} \cdot \inf_{\substack{x\in L_m\\ x\neq 0}} \frac{\|Tx|Y_1\|^{\Theta}}{\|x|X\|^{\Theta}} \\ &\leq C \sup_{\substack{x\in M_n\\ x\neq 0}} \frac{\|Tx|Y_0\|^{1-\Theta}}{\|x|X\|^{1-\Theta}} \cdot b_m^{\Theta}(T : X \to Y_1).\end{aligned}$$

Since M_n is arbitrary we have

$$\inf_{\substack{x\in L_{m+n-1}\\ x\neq 0}} \frac{\|Tx|Y\|}{\|x|X\|} \leq C\, c_n^{1-\Theta}(T : X \to Y_0) \cdot b_m^{\Theta}(T : X \to Y_1).$$

Now take supremum with respect to L_{m+n-1} we get the desired result. ∎

There is an interesting relation of Weyl numbers and absolutely $(r,2)$-summing norms. Let $2 \leq r < \infty$. An operator $T \in \mathcal{L}(X,Y)$ is said to be absolutely $(r,2)$-summing if there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$ the inequality

$$\Big(\sum_{j=1}^{n} \|\, Tx_j\, |Y\|^r\Big)^{1/r} \leq C \sup_{x'\in X', \|x'|X'\|\leq 1} \Big(\sum_{j=1}^{n} |\langle x_j, x'\rangle|^2\Big)^{1/2} \tag{4.7}$$

holds, see [84, Chapter 17] or [87, Section 1.2]. Recall that X' refers to the dual space of X. The norm $\pi_{r,2}(T)$ is given by the infimum with respect to $C > 0$ satisfying (4.7). The class of all those operators is denoted by $\mathcal{B}_{r,2}(X,Y)$. In the literature sometimes the notion $P_{r,2}(T)$ is used instead of $\pi_{r,2}(T)$. If $r = 2$ we simply write $\pi_2(T)$. The announced relation between Weyl numbers and the $(r,2)$-summing norms is given by the following lemma, see [85].

Lemma 4.14. *Let X, Y be Banach spaces. Let $2 \leq r < \infty$ and $T \in \mathcal{B}_{r,2}(X,Y)$. Then for any $n \in \mathbb{N}$ we have*

$$x_n(T) \leq n^{-\frac{1}{r}} \pi_{r,2}(T).$$

Remark 4.15. There is an analogue for Bernstein numbers in case $r = 2$, i.e., if T is 2-summing operator then

$$b_n(T) \leq n^{-\frac{1}{2}} \pi_2(T)$$

holds for all $n \in \mathbb{N}$. For a proof we refer to Pietsch [89].

Finally we turn to the asymptotic behavior of Weyl and Bernstein numbers of the embedding $id_{p_0,p}^m : \ell_{p_0}^m \to \ell_p^m$ which is needed for our proof. The results for Weyl numbers have been obtained at various places, we refer to Lubitz [65], König [55, Section 3.c], Caetano [13, 14] and Zhang, Fang, Huang [145]. In the case of Bernstein numbers we refer to [40] and [75].

Lemma 4.16. (a) *Let $m, n \in \mathbb{N}$ and $2n \le m$. Then we have*

$$x_n(id^m_{p_0,p}) \asymp \begin{cases} 1 & \text{if } \ 2 \le p_0 \le p \le \infty, & (4.8a) \\ n^{\frac{1}{p}-\frac{1}{p_0}} & \text{if } \ 0 < p_0 \le p \le 2, & (4.8b) \\ n^{\frac{1}{2}-\frac{1}{p_0}} & \text{if } \ 0 < p_0 \le 2 \le p \le \infty, & (4.8c) \\ m^{\frac{1}{p}-\frac{1}{p_0}} & \text{if } \ 0 < p < p_0 \le 2. & (4.8d) \end{cases}$$

(b) *Let $2 \le p < p_0 \le \infty$ and $n, m, k \in \mathbb{N}$, $k \ge 2$. Then we have*

(i) $x_n(id^m_{p_0,p}) \lesssim \Big(\dfrac{m}{n}\Big)^{1/r}$ *if* $n \le m$, $\dfrac{1}{r} = \dfrac{1/p - 1/p_0}{1 - 2/p_0}$,

(ii) $x_n(id^{kn}_{p_0,p}) \asymp 1$.

(c) *Let $0 < p \le 2 < p_0 \le \infty$ and $n, m \in \mathbb{N}$. Then*

(i) $x_n(id^m_{p_0,p}) \gtrsim m^{\frac{1}{p}-\frac{1}{2}}$ *if* $n \le \frac{m}{2}$,

(ii) $x_n(id^m_{p_0,p}) \gtrsim m^{\frac{1}{p}-\frac{1}{p_0}}$ *if* $n \le m^{\frac{2}{p_0}}$.

Lemma 4.17. (i) *Let $1 \le p_0, p \le \infty$ and $n \in \mathbb{N}$. It holds*

$$b_n(id^{2n}_{p_0,p}) \gtrsim \begin{cases} 1 & \text{if } \ 2 \le p \le p_0, & (4.9a) \\ n^{\frac{1}{p}-\frac{1}{2}} & \text{if } \ p \le 2 \le p_0, & (4.9b) \\ n^{\frac{1}{p}-\frac{1}{p_0}} & \text{if } \ p_0 \le p \ \text{ or } \ p \le p_0 \le 2. & (4.9c) \end{cases}$$

(ii) *Let $1 < p \le \max(p, 2) < p_0 \le \infty$ and $n, m \in \mathbb{N}$. It holds*

$$b_n(id^m_{p_0,p}) \gtrsim m^{\frac{1}{p}-\frac{1}{p_0}}, \quad 1 \le n \le \big[m^{\frac{2}{p_0}}\big].$$

4.2 Wavelets

For us it will be convenient to consider the wavelet characterization of the spaces of dominating mixed smoothness. Let $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ and $m = (m_1, \ldots, m_d) \in \mathbb{Z}^d$. Then we put

$$Q_{\nu,m} := \Big\{x \in \mathbb{R}^d : \ 2^{-\nu_\ell}\, m_\ell < x_\ell < 2^{-\nu_\ell}\,(m_\ell + 1)\,,\ \ell = 1, \ldots, d\Big\}.$$

By $\chi_{\nu,m}$ we denote the characteristic function of $Q_{\nu,m}$. First we have to introduce some sequence spaces.

Definition 4.18. If $0 < p, q \le \infty$, $t \in \mathbb{R}$ and $\lambda := \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0^d,\ m \in \mathbb{Z}^d\}$, then we define

$$s^t_{p,q}b := \Big\{\lambda : \|\lambda|s^t_{p,q}b\| = \Big(\sum_{\nu\in\mathbb{N}_0^d} 2^{|\nu|_1(t-\frac{1}{p})q}\Big(\sum_{m\in\mathbb{Z}^d} |\lambda_{\nu,m}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}$$

and, if $p < \infty$,

$$s^t_{p,q}f = \Big\{\lambda : \|\lambda|s^t_{p,q}f\| = \Big\|\Big(\sum_{\nu\in\mathbb{N}_0^d}\sum_{m\in\mathbb{Z}^d} |2^{|\nu|_1 t}\lambda_{\nu,m}\chi_{\nu,m}(\cdot)|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| < \infty\Big\}$$

with the usual modification for p or/and q equal to ∞.

Remark 4.19. Let $\sigma \in \mathbb{R}$. For later use we mention that the mapping

$$J_\sigma : \ \{\lambda_{\nu,m}\}_{\nu,m} \mapsto \{2^{\sigma|\nu|_1} \lambda_{\nu,m}\}_{\nu,m} \tag{4.10}$$

yields an isomorphism of $s^t_{p,q}a$ onto $s^{t-\sigma}_{p,q}a$, $a \in \{b, f\}$.

Now we recall wavelet bases of Besov and Lizorkin-Triebel spaces of dominating mixed smoothness. Let $N \in \mathbb{N}$. Then there exists $\psi_0, \psi_1 \in C^N(\mathbb{R})$, compactly supported,

$$\int_{-\infty}^{\infty} \xi^r \psi_1(\xi) \, \mathrm{d}\xi = 0 \,, \qquad r = 0, 1, \ldots, N \,,$$

such that $\{2^{j/2} \psi_{j,m} : \ j \in \mathbb{N}_0, \ m \in \mathbb{Z}\}$, where

$$\psi_{j,m}(\xi) := \begin{cases} \psi_0(\xi - m) & \text{if} \quad j = 0, \ m \in \mathbb{Z} \,, \\ \sqrt{1/2}\, \psi_1(2^{j-1}\xi - m) & \text{if} \quad j \in \mathbb{N} \,, \ m \in \mathbb{Z} \,, \end{cases}$$

is an orthonormal basis in $L_2(\mathbb{R})$, see [141]. By putting

$$\Psi_{\nu,m}(x) := \prod_{i=1}^{d} \psi_{\nu_i,m_i}(x_i) \,, \qquad \nu \in \mathbb{N}_0^d, \ m \in \mathbb{Z}^d \,,$$

we obtain a tensor product wavelet basis of $L_2(\mathbb{R}^d)$. Vybiral [140, Theorem 2.12] has proved the following.

Lemma 4.20. *Let $0 < p, q \le \infty$ and $t \in \mathbb{R}$.*
(i) *There exists $N = N(t, p) \in \mathbb{N}$ such that the mapping*

$$\mathcal{W} : \quad f \mapsto \big\{2^{|\nu|_1} \langle f, \Psi_{\nu,m}\rangle\big\}_{\nu \in \mathbb{N}_0^d,\, m \in \mathbb{Z}^d}$$

is an isomorphism of $S^t_{p,q}B(\mathbb{R}^d)$ onto $s^t_{p,q}b$.
(ii) *Let $0 < p < \infty$. Then there exists $N = N(t, p, q) \in \mathbb{N}$ such that the mapping $\mathcal{W}$ is an isomorphism of $S^t_{p,q}F(\mathbb{R}^d)$ onto $s^t_{p,q}f$.*

Remark 4.21. Since the functions $\Psi_{\nu,m}$, $\nu \in \mathbb{N}_0^d$, $m \in \mathbb{Z}^d$, do not belong to $\mathcal{S}(\mathbb{R}^d)$ we need further explanations. For $\varepsilon > 0$ we have $S^t_{p,q}A(\mathbb{R}^d) \hookrightarrow S^{t-\varepsilon}_{p,p}B(\mathbb{R}^d)$. We can choose N large enough such that $\Psi_{\nu,m} \in [S^{t-\varepsilon}_{p,p}B(\mathbb{R}^d)]'$. Hence we may interpret $\Psi_{\nu,m}$ as a bounded linear functional on $S^{t-\varepsilon}_{p,p}B(\mathbb{R}^d)$ and $\langle f, \Psi_{\nu,m}\rangle$ is the value of this functional at f. Vice versa f also can be interpreted as a linear functional on a Besov space containing $\Psi_{\nu,m}$. We refer to [140, Section 2.4] for more details.

For technical reasons we need a few more sequence spaces. Let t, p and q be fixed. Let the wavelet basis $\{\Psi_{\nu,m}\}_{\nu \in \mathbb{N}_0^d,\, m \in \mathbb{Z}^d}$ be admissible in the sense of Lemma 4.20. We put

$$A^\Omega_\nu := \Big\{m \in \mathbb{Z}^d : \quad \operatorname{supp} \Psi_{\nu,m} \cap \Omega \neq \emptyset\Big\} \,, \qquad \nu \in \mathbb{N}_0^d \,. \tag{4.11}$$

Definition 4.22. If $0 < p,q \le \infty$, $t \in \mathbb{R}$ and

$$\lambda = \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0^d,\ m \in A_\nu^\Omega\}\,,$$

then we define

$$s^{t,\Omega}_{p,q}b := \Big\{\lambda :\ \|\lambda|s^{t,\Omega}_{p,q}b\| = \Big(\sum_{\nu\in\mathbb{N}_0^d} 2^{|\nu|_1(t-\frac{1}{p})q}\Big(\sum_{m\in A_\nu^\Omega} |\lambda_{\nu,m}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}$$

and, if $p < \infty$,

$$s^{t,\Omega}_{p,q}f := \Big\{\lambda :\ \|\lambda|s^{t,\Omega}_{p,q}f\| = \Big\|\Big(\sum_{\nu\in\mathbb{N}_0^d}\sum_{m\in A_\nu^\Omega} |2^{|\nu|_1 t}\lambda_{\nu,m}\chi_{\nu,m}(\cdot)|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| < \infty\Big\}.$$

In addition we need the following sequence of subspaces.

Definition 4.23. If $0 < p,q \le \infty$, $t \in \mathbb{R}$, $\mu \in \mathbb{N}_0$ and

$$\lambda = \{\lambda_{\nu,m} \in \mathbb{C} : \nu \in \mathbb{N}_0^d,\ |\nu|_1 = \mu,\ m \in A_\nu^\Omega\}\,,$$

then we define

$$(s^{t,\Omega}_{p,q}b)_\mu = \Big\{\lambda : \|\lambda|(s^{t,\Omega}_{p,q}b)_\mu\| = \Big(\sum_{|\nu|_1=\mu} 2^{|\nu|_1(t-\frac{1}{p})q}\Big(\sum_{m\in A_\nu^\Omega} |\lambda_{\nu,m}|^p\Big)^{q/p}\Big)^{1/q} < \infty\Big\}$$

and, if $p < \infty$,

$$(s^{t,\Omega}_{p,q}f)_\mu = \Big\{\lambda : \|\lambda|(s^{t,\Omega}_{p,q}f)_\mu\| = \Big\|\Big(\sum_{|\nu|_1=\mu}\sum_{m\in A_\nu^\Omega} |2^{|\nu|_1 t}\lambda_{\nu,m}\chi_{\nu,m}(\cdot)|^q\Big)^{1/q}\Big|L_p(\mathbb{R}^d)\Big\| < \infty\Big\}.$$

To avoid repetitions we shall use $s^t_{p,q}a$, $s^{t,\Omega}_{p,q}a$, $(s^{t,\Omega}_{p,q}a)_\mu$ with $a \in \{b,f\}$ in case that an assertion holds for both scales simultaneously. In this section we do not deal with the spaces $s^{t,\Omega}_{\infty,q}f$ and $(s^{t,\Omega}_{\infty,q}f)_\mu$ except $s^{t,\Omega}_{\infty,\infty}f := s^{t,\Omega}_{\infty,\infty}b$ and $(s^{t,\Omega}_{\infty,\infty}f)_\mu := (s^{t,\Omega}_{\infty,\infty}b)_\mu$. Hence whenever we write $s^{t,\Omega}_{\infty,q}a$ or $(s^{t,\Omega}_{\infty,q}a)_\mu$, this has to be interpreted as $s^{t,\Omega}_{\infty,q}b$ and $(s^{t,\Omega}_{\infty,q}b)_\mu$. The two following elementary lemmas are taken from [140, Lemma 3.10] and [45, Lemma 6.4.2].

Lemma 4.24. (i) *We have*

$$|A_\nu^\Omega| \asymp 2^{|\nu|_1}, \qquad D_\mu := \sum_{|\nu|_1=\mu} |A_\nu^\Omega| \asymp \mu^{d-1}2^\mu$$

with equivalence constants independent of $\nu \in \mathbb{N}_0^d$ *and* $\mu \in \mathbb{N}_0$.
(ii) *Let* $0 < p < \infty$ *and* $t \in \mathbb{R}$. *Then*

$$s^{t,\Omega}_{p,p}f = s^{t,\Omega}_{p,p}b$$

and

$$(s^{t,\Omega}_{p,p}f)_\mu = (s^{t,\Omega}_{p,p}b)_\mu = 2^{\mu(t-\frac{1}{p})}\ell_p^{D_\mu}, \quad \mu \in \mathbb{N}_0\,,$$

with the obvious interpretation for the quasi-norms.

Lemma 4.25. (i) *Let $0 < p_0, p, q \le \infty$ and $t \in \mathbb{R}$. Then*

$$\| \, id^*_\mu \, : (s^{t,\Omega}_{p_0,q}a)_\mu \to (s^{t,\Omega}_{p,q}a)_\mu \| \asymp 2^{\mu(\frac{1}{p_0}-\frac{1}{p})_+}$$

with equivalence constants independent of $\mu \in \mathbb{N}_0$.
(ii) *Let $0 < p, q_0, q \le \infty$ and $t \in \mathbb{R}$. Then*

$$\| id^*_\mu : (s^{t,\Omega}_{p,q_0}a)_\mu \to (s^{t,\Omega}_{p,q}a)_\mu \| \asymp \mu^{(d-1)(\frac{1}{q}-\frac{1}{q_0})_+}$$

with equivalence constants independent of $\mu \in \mathbb{N}_0$.

Corollary 4.26. *Let $0 < p_0, p, q_0, q \le \infty$ and $t \in \mathbb{R}$. Then*

$$\| \, id^*_\mu \, : (s^{t,\Omega}_{p_0,q_0}a)_\mu \to (s^{0,\Omega}_{p,q}a)_\mu \| \lesssim 2^{\mu\left(-t+(\frac{1}{p_0}-\frac{1}{p})_+\right)} \mu^{(d-1)(\frac{1}{q}-\frac{1}{q_0})_+},$$

with a constant behind $\lesssim$ independent of μ.

Proof. This is an immediate consequence of Lemma 4.25. ■

Sometimes the previous estimate can be improved. The proof of lemma below follows similarly in [45, page 158].

Lemma 4.27. *Let $0 < p_0 < p < \infty$, $0 < q_0, q \le \infty$ and $t \in \mathbb{R}$. Then*

$$\| id^*_\mu : (s^{t,\Omega}_{p_0,q_0}f)_\mu \to (s^{0,\Omega}_{p,q}f)_\mu \| \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}.$$

Proof. Let λ be a sequence such that $\lambda_{\nu,m} = 0$ if $|\nu|_1 \ne \mu$. Since $p_0 < p$ the Sobolev-type embedding yields

$$s^{t,\Omega}_{p_0,q_0}f \hookrightarrow s^{t-\frac{1}{p_0}+\frac{1}{p},\Omega}_{p,q}f\,,$$

see Lemma 1.34. From this we have

$$\begin{aligned} \|\lambda|(s^{0,\Omega}_{p,q}f)_\mu\| &= \|\lambda|s^{0,\Omega}_{p,q}f\| = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\|\lambda|s^{t-\frac{1}{p_0}+\frac{1}{p},\Omega}_{p,q}f\| \\ &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\|\lambda|s^{t,\Omega}_{p_0,q_0}f\| = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\|\lambda|(s^{t,\Omega}_{p_0,q_0}f)_\mu\|\,. \end{aligned}$$

This proves the claim. ■

4.3 Decomposition method

Tensor product of Besov and Sobolev spaces are special cases of Triebel-Lizorkin spaces $S^t_{p_0,q}F(\Omega)$ with $q \in \{2, p_0\}$. By means of wavelet characterizations we switch from the consideration of $\omega_n\big(id : S^t_{p_0,q}F(\Omega) \to L_p(\Omega)\big)$ to $\omega_n\big(id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f\big)$. To estimate the upper bound of Bernstein numbers we employ the inequality

$$b_{2n-1}(T) \le \min\Big\{e\Big(\prod_{k=1}^n x_k(T)\Big)^{1/n},\ 2\sqrt{2}e_{2n-1}(T)\Big\},$$

see Lemmas 4.8 and 4.10. Concerning the estimate from above of Weyl numbers, the main idea of our proof is the splitting of id^* into a sum of identities between building blocks. And then we employ the additivity property of Weyl numbers to trace all back to the estimate of $x_{n_\mu}(id : \ell_r^m \to \ell_s^m)$ with appropriate $m, n_\mu \in \mathbb{N}$ and $0 < r,s \le \infty$. Let us mention that a similar splitting has been used by Vybiral [140, Chapter 3] for the estimates of related entropy numbers.

To consider the embedding $id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f$ we assume that p_0 varies in $(0,\infty]$ and p in $(0,\infty)$. We split

$$id^* = \sum_{\mu=0}^{\infty} id_\mu = \sum_{\mu=0}^{J} id_\mu + \sum_{\mu=J+1}^{L} id_\mu + \sum_{\mu=L+1}^{\infty} id_\mu, \tag{4.12}$$

where

$$id_\mu : \; s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f$$

with

$$(id_\mu\lambda)_{\nu,m} := \begin{cases} \lambda_{\nu,m} & \text{if } |\nu|_1 = \mu, \\ 0 & \text{otherwise} \end{cases}$$

and J and L are at our disposal. These numbers J and L will be chosen in dependence on the parameters. We observe that for $n \in \mathbb{N}$ and $\mu \in \mathbb{N}_0$ we have

$$x_n\big(id_\mu : \; s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f\big) = x_n\big(id^*_\mu : \; (s^{t,\Omega}_{p_0,q}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big), \tag{4.13}$$

in particular, $\|id_\mu\| = \|id^*_\mu\|$. The additivity, the monotonicity of the Weyl numbers and the quasi-triangle inequality (4.1) yield

$$\begin{aligned} x^\rho_n(id^*) &\le \sum_{\mu=0}^{J} x^\rho_{n_\mu}(id_\mu) + \sum_{\mu=J+1}^{L} x^\rho_{n_\mu}(id_\mu) + \sum_{\mu=L+1}^{\infty} \|id_\mu\|^\rho, \qquad \rho = \min(1,p) \\ &= \sum_{\mu=0}^{J} x^\rho_{n_\mu}(id^*_\mu) + \sum_{\mu=J+1}^{L} x^\rho_{n_\mu}(id^*_\mu) + \sum_{\mu=L+1}^{\infty} \|id^*_\mu\|^\rho, \end{aligned} \tag{4.14}$$

where $n-1 = \sum_{\mu=0}^{L}(n_\mu - 1)$, see (4.13). Here we used the fact that the spaces $s^{t,\Omega}_{p,q}a$ (and also $s^t_{p,q}a$, $S^t_{p,q}A(\mathbb{R}^d)$) are ρ-Banach spaces with $\rho = \min(1,p,q)$. By Corollary 4.26, we have

$$\|id^*_\mu : (s^{t,\Omega}_{p_0,q}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\| \lesssim 2^{-\mu\left(t-(\frac{1}{p_0}-\frac{1}{p})_+\right)}\mu^{(d-1)(\frac{1}{2}-\frac{1}{q})_+},$$

which results in the estimate

$$\sum_{\mu=L+1}^{\infty} \|id^*_\mu\|^\rho \lesssim 2^{-L\rho\left(t-(\frac{1}{p_0}-\frac{1}{p})_+\right)} L^{(d-1)\rho(\frac{1}{2}-\frac{1}{q})_+}\,. \tag{4.15}$$

Now we choose $n_\mu := D_\mu + 1,\ \mu = 0,1,\ldots,J$. Then we get

$$\sum_{\mu=0}^{J} n_\mu \asymp \sum_{\mu=0}^{J} \mu^{(d-1)}2^\mu \asymp J^{d-1}2^J \tag{4.16}$$

and $x_{n_\mu}(id^*_\mu) = 0$ for $\mu = 0,1,\ldots,J$, see property (d) of the s-numbers, which implies

$$\sum_{\mu=0}^{J} x^\rho_{n_\mu}(id^*_\mu) = 0\,. \tag{4.17}$$

Summarizing (4.14), (4.15) and (4.17) we have found

$$x_n^\rho(id^*) \lesssim \sum_{\mu=J+1}^{L} x_{n_\mu}^\rho(id_\mu^*) + 2^{-L\rho\left(t-(\frac{1}{p_0}-\frac{1}{p})_+\right)} L^{(d-1)\rho(\frac{1}{2}-\frac{1}{q})_+} . \tag{4.18}$$

For the estimate from below we use the following lemma. We recall that the notation $id_{p_0,p}^m$ refers to the identity $id_{p_0,p}^m : \ \ell_{p_0}^m \to \ell_p^m$.

Lemma 4.28. *For all $\mu \in \mathbb{N}_0$ and all $n \in \mathbb{N}$ we have*

$$\max\left\{2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\,\omega_n(id_{p_0,p}^{A_\mu})\,,\omega_n(id_\mu^*)\right\} \le \omega_n(id^*)\,. \tag{4.19}$$

Here $A_\mu = |A_\nu^\Omega|$ for some ν with $|\nu|_1 = \mu$.

Proof. *Step 1.* We consider the following diagram

$$\begin{array}{ccc} s_{p_0,q}^{t,\Omega}f & \xrightarrow{id^*} & s_{p,2}^{0,\Omega}f \\ {\scriptstyle id^1}\Big\uparrow & & \Big\downarrow{\scriptstyle id^2} \\ (s_{p_0,q}^{t,\Omega}f)_\mu & \xrightarrow{id_\mu^*} & (s_{p,2}^{0,\Omega}f)_\mu\,. \end{array}$$

Here id^1 is the canonical embedding and id^2 is the canonical projection. Since

$$id_\mu^* = id_2 \circ id^* \circ id_1$$

the property (c) of the s-numbers yields

$$\omega_n(id_\mu^*) \le \|\, id^1 \,\|\, \|\, id^2 \,\|\, \omega_n(id^*) = \omega_n(id^*)\,.$$

Step 2. We consider the following commutative diagram

$$\begin{array}{ccc} s_{p_0,q}^{t,\Omega}f & \xrightarrow{id^*} & s_{p,2}^{0,\Omega}f \\ {\scriptstyle id^1}\Big\uparrow & & \Big\downarrow{\scriptstyle id^2} \\ 2^{\mu(t-\frac{1}{p_0})}\ell_{p_0}^{A_\mu} & \xrightarrow{I_\mu} & 2^{\mu(0-\frac{1}{p})}\ell_p^{A_\mu}\,. \end{array}$$

Here id^1 is the canonical embedding, whereas id^2 is the canonical projection. From property (c) of the s-numbers we derive

$$\omega_n(I_\mu) = \omega_n(id^2 \circ id^* \circ id^1) \le \|id^1\|\,\|id^2\|\,\omega_n(id^*) = \omega_n(id^*)\,.$$

Again property (c) of the s-numbers guarantees

$$\omega_n(I_\mu) = 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\,\omega_n(id_{p_0,p}^{A_\mu})\,.$$

The proof is complete. ∎

Now we turn to the problem of reducing the estimates of Weyl and Bernstein numbers of the identity $id_\mu^* : \ (s_{p_0,q}^{t,\Omega}f)_\mu \to (s_{p,2}^{0,\Omega}f)_\mu$ to the estimates of corresponding numbers of $id : \ell_r^{D_\mu} \to \ell_s^{D_\mu}$ for some r and s. The following lemmas hold for both, Weyl and Bernstein numbers.

Proposition 4.29. *Let $0 < p_0 \le \infty$ and $t \in \mathbb{R}$.*
(i) *If $0 < p \le 2$ and $p \le \delta \le 2$, then*

$$\begin{aligned} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, \omega_n(id^{D_\mu}_{p_0,p}) &\lesssim \omega_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,p_0}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \\ &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\delta})}\, \omega_n(id^{D_\mu}_{p_0,\delta}). \end{aligned} \tag{4.20}$$

(ii) *If $2 \le p < \infty$ and $2 \le \gamma \le p$, then*

$$\begin{aligned} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\gamma})}\, \omega_n(id^{D_\mu}_{p_0,\gamma}) &\lesssim \omega_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,p_0}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \\ &\lesssim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, \omega_n(id^{D_\mu}_{p_0,p})\,. \end{aligned} \tag{4.21}$$

(iii) *If $0 < p < \infty$ and $0 < \varepsilon < p$, then*

$$\omega_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,p_0}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, \omega_n(id^{D_\mu}_{p_0,p-\varepsilon})\,. \tag{4.22}$$

Proof. *Step 1.* We prove the right-hand sides of parts (i) and (ii). We put $\delta_0 := \delta$ if $p \le 2$ and $\delta_0 = p$ if $p \ge 2$. By considering the following chain of embeddings

$$(s^{t,\Omega}_{p_0,p_0}f)_\mu \xrightarrow{id^2} (s^{0,\Omega}_{\delta_0,\delta_0}f)_\mu \xrightarrow{id^1} (s^{0,\Omega}_{p,2}f)_\mu\,,$$

and using property (c) of the s-numbers we conclude

$$\omega_n(id^*_\mu) = \omega_n(id^1 \circ id^2) \le \|\, id^1 \,\|\, \omega_n(id^2)\,.$$

Corollary 4.26 yields

$$\|id^1\| \lesssim 2^{\mu(\frac{1}{\delta_0}-\frac{1}{p})_+} \mu^{(d-1)(\frac{1}{2}-\frac{1}{\delta_0})_+} = \mu^{(d-1)(\frac{1}{2}-\frac{1}{\delta_0})_+}\,.$$

From Lemma 4.24 (ii), we derive

$$\omega_n(id^2) \asymp 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\delta_0})}\, \omega_n(id^{D_\mu}_{p_0,\delta_0})\,,$$

Altogether this implies

$$\omega_n(id^*_\mu) \lesssim \mu^{(d-1)(\frac{1}{2}-\frac{1}{\delta_0})_+}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\delta_0})}\, \omega_n(id^{D_\mu}_{p_0,\delta_0})\,.$$

Step 2. We prove the left-hand sides of parts (i) and (ii). We define $\gamma_0 := p$ if $p \le 2$ and $\gamma_0 := \gamma$ if $p \ge 2$. This time we employ following chain of embeddings

$$(s^{t,\Omega}_{p_0,p_0}f)_\mu \xrightarrow{id^*_\mu} (s^{0,\Omega}_{p,2}f)_\mu \xrightarrow{id^1} (s^{0,\Omega}_{\gamma_0,\gamma_0}f)_\mu$$

to obtain

$$\omega_n(id^1 \circ id^*_\mu) \le \|\, id^1 \,\|\, \omega_n(id^*_\mu).$$

By Corollary 4.26, we get

$$\|\, id^1 \,\| \lesssim 2^{\mu(\frac{1}{p}-\frac{1}{\gamma_0})_+} \mu^{(d-1)(\frac{1}{\gamma_0}-\frac{1}{2})_+} = \mu^{(d-1)(\frac{1}{\gamma_0}-\frac{1}{2})_+}\,.$$

Lemma 4.24 (ii) yields

$$\omega_n(id^1 \circ id^*_\mu) = \omega_n\big(id : (s^{t,\Omega}_{p_0,p_0}f)_\mu \to (s^{0,\Omega}_{\gamma_0,\gamma_0}f)_\mu\big) \asymp 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\gamma_0})}\, \omega_n(id^{D_\mu}_{p_0,\gamma_0})\,.$$

Inserting this in our previous estimate we find

$$\omega_n(id^*_\mu) \gtrsim \mu^{-(d-1)(\frac{1}{\gamma_0}-\frac{1}{2})_+}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{\gamma_0})}\, \omega_n(id^{D_\mu}_{p_0,\gamma_0})$$

which proves the claims.
Step 3. Proof of (iii). We consider the following chain of embeddings

$$(s^{t,\Omega}_{p_0,p_0}f)_\mu \xrightarrow{id^2} (s^{r,\Omega}_{p-\varepsilon,p-\varepsilon}f)_\mu \xrightarrow{id^1} (s^{0,\Omega}_{p,2}f)_\mu$$

Clearly,

$$\omega_n(id^*_\mu) = \omega_n(id^1 \circ id^2) \le \|\, id^1 \,\|\, \omega_n(id^2)$$

and by Lemma 4.27 we have

$$\|\, id^1 \,\| \lesssim 2^{\mu(-r+\frac{1}{p-\varepsilon}-\frac{1}{p})}\,.$$

Further we know

$$\omega_n(id^2) \asymp 2^{\mu(r-\frac{1}{p-\varepsilon}-t+\frac{1}{p_0})}\, \omega_n(id^{D_\mu}_{p_0,p-\varepsilon})\,.$$

This is enough to establish (4.22). ■

Proposition 4.30. *Let $t \in \mathbb{R}$ and $0 < p_0, p < \infty$.*
(i) *If $0 < p_0 \le 2$, then we have*

$$\mu^{-(d-1)(\frac{1}{p}-\frac{1}{2})_+} 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})}\omega_n(id^{D_\mu}_{2,p}) \lesssim \omega_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big). \tag{4.23}$$

(ii) *If $\varepsilon > 0$ such that $p_0 - \varepsilon > 0$, then*

$$\mu^{-(d-1)(\frac{1}{p}-\frac{1}{2})_+} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\omega_n(id^{D_\mu}_{p_0-\varepsilon,p}) \lesssim \omega_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big). \tag{4.24}$$

(ii) *If $2 \le p < \infty$, then*

$$\omega_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \lesssim \mu^{(d-1)(\frac{1}{p_0}-\frac{1}{2})_+} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\omega_n(id^{D_\mu}_{p_0,2}). \tag{4.25}$$

Proof. *Step 1.* Proof of (i). We consider the following diagram

$$\begin{array}{ccc} (s^{t,\Omega}_{p_0,2}f)_\mu & \xrightarrow{id^*_\mu} & (s^{0,\Omega}_{p,2}f)_\mu \\ {\scriptstyle id_1}\big\uparrow & & \big\downarrow{\scriptstyle id_3} \\ (s^{t,\Omega}_{2,2}f)_\mu & \xrightarrow{id_2} & (s^{0,\Omega}_{p,p}f)_\mu \end{array}$$

and obtain

$$\omega_n(id_2) \;\lesssim\; \|id_1\| \cdot \|id_3\| \cdot \omega_n(id^*_\mu). \tag{4.26}$$

By Lemmas 4.25 (i) and 4.24 (ii) we have

$$\|id_1\| \lesssim 1\,, \qquad \|id_3\| \lesssim \mu^{(d-1)(\frac{1}{p}-\frac{1}{2})_+}$$

and

$$\omega_n(id_2) \asymp 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})}\omega_n(id^{D_\mu}_{2,p})\,.$$

This together with (4.26) results in (4.23).
Step 2. Proof of (ii). We consider the following diagram

$$\begin{array}{ccc} (s^{t,\Omega}_{p_0,2}f)_\mu & \xrightarrow{id^*_\mu} & (s^{0,\Omega}_{p,2}f)_\mu \\ {\scriptstyle id_1}\Big\uparrow & & \Big\downarrow{\scriptstyle id_3} \\ (s^{0,\Omega}_{p_0-\varepsilon,p_0-\varepsilon}f)_\mu & \xrightarrow{id_2} & (s^{0,\Omega}_{p,p}f)_\mu\,. \end{array}$$

Property (c) yields

$$\omega_n(id_2) \ \le\ \|id_1\|\cdot\|id_3\|\cdot\omega_n(id^*_\mu).$$

This together with

$$\|id_1\| \lesssim 2^{\mu(t+\frac{1}{p_0-\varepsilon}-\frac{1}{p_0})}\,, \qquad \|id_3\| \lesssim \mu^{(d-1)(\frac{1}{p}-\frac{1}{2})_+},$$

see Lemma 4.25, and

$$\omega_n(id_2) \asymp 2^{\mu(-\frac{1}{p}+\frac{1}{p_0-\varepsilon})}\omega_n(id^{D_\mu}_{p_0-\varepsilon,p}),$$

see Lemma 4.24, claims the estimate.
Step 3. Proof of (iii). This time we consider the following diagram

$$\begin{array}{ccc} (s^{t,\Omega}_{p_0,p_0}f)_\mu & \xrightarrow{id_2} & (s^{0,\Omega}_{2,2}f)_\mu \\ {\scriptstyle id_1}\Big\uparrow & & \Big\downarrow{\scriptstyle id_3} \\ (s^{t,\Omega}_{p_0,2}f)_\mu & \xrightarrow{id^*_\mu} & (s^{0,\Omega}_{p,2}f)_\mu \end{array}$$

and obtain

$$\omega_n(id^*_\mu) \ \le\ \|id_1\|\cdot\|id_3\|\cdot\omega_n(id_2)\,. \tag{4.27}$$

Employing Lemmas 4.25 and 4.24 we have

$$\|id_1\| \lesssim \mu^{(d-1)(\frac{1}{p_0}-\frac{1}{2})_+}\,, \qquad \|id_3\| \lesssim 2^{\mu(\frac{1}{2}-\frac{1}{p})}$$

and

$$\omega_n(id_2) \asymp 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})}\omega_n(id^{D_\mu}_{p_0,2}).$$

From this and (4.27) the claim follows. The proof is complete. ■

Remark 4.31. Since we only use property (c) of s-numbers in the proof, Propositions 4.29 and 4.30 hold true for any s-numbers.

Lemma 4.32. (i) *Let $1 < p \le 2 < p_0 < \infty$. Then we have*

$$b_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \ \gtrsim\ 2^{-t\mu}, \quad n = \big[\mu^{d-1}2^{\frac{2\mu}{p_0}}\big].$$

(ii) *Let $0 < p \le 2 < p_0 < \infty$. Then we have*

$$x_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \ \gtrsim\ 2^{-t\mu}, \quad n = \big[\mu^{d-1}2^{\frac{2\mu}{p_0}}\big].$$

Proof. *Step 1.* We prove (i). Since $p \le 2 < p_0$ we have the chain of embeddings

$$(s^{t,\Omega}_{p_0,2}b)_\mu \xrightarrow{id^1} (s^{t,\Omega}_{p_0,2}f)_\mu \xrightarrow{id^*_\mu} (s^{0,\Omega}_{p,2}f)_\mu \xrightarrow{id^2} (s^{0,\Omega}_{p,2}b)_\mu\,, \tag{4.28}$$

with the norms of id^1 and id^2 independent of μ see [45, Lemma 5.3.4], see also Lemma 1.32. From the definition of Bernstein numbers and (4.28) we deduce the existence of some constant $C>0$ such that

$$\begin{aligned} b_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) &= \sup_{L_n}\inf_{\lambda\in L_n}\frac{\|\lambda\,|\,(s^{0,\Omega}_{p,2}f)_\mu\|}{\|\lambda\,|\,(s^{t,\Omega}_{p_0,2}f)_\mu\|}\\ &\ge C\sup_{L_n}\inf_{\lambda\in L_n}\frac{\|\lambda\,|\,(s^{0,\Omega}_{p,2}b)_\mu\|}{\|\lambda\,|\,(s^{t,\Omega}_{p_0,2}b)_\mu\|}\,, \end{aligned} \tag{4.29}$$

where C is independent of n. Recall that the supremum is taken over all linear subspaces L_n of dimension $n=\big[\mu^{d-1}2^{\frac{2\mu}{p_0}}\big]$ in $(s^{0,\Omega}_{2,2}f)_\mu$. Note that

$$\frac{\|\lambda\,|\,(s^{0,\Omega}_{p,2}b)_\mu\|}{\|\lambda\,|\,(s^{t,\Omega}_{p_0,2}b)_\mu\|} = \frac{2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\Big(\sum\limits_{|\nu|_1=\mu}\Big(\sum\limits_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^p\Big)^{2/p}\Big)^{1/2}}{\Big(\sum\limits_{|\nu|_1=\mu}\Big(\sum\limits_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^{p_0}\Big)^{2/p_0}\Big)^{1/2}}. \tag{4.30}$$

We put $\Delta_\mu=\{\nu\in\mathbb{N}^d_0\ :\ |\nu|_1=\mu\}$. For each $\nu\in\Delta_\mu$ the inequality

$$b_k(id^{|A^\Omega_\nu|}_{p_0,p}) \gtrsim 2^{|\nu|_1(\frac{1}{p}-\frac{1}{p_0})},\quad k=\big[2^{|\nu|_1\frac{2}{p_0}}\big], \tag{4.31}$$

see Lemma 4.17 (ii), implies that there exists a linear subspace L^ν_k in $\mathbb{R}^{|A^\Omega_\nu|}\times\mathbb{R}^{|\Delta_\mu|}$ of dimension $k=\big[2^{|\nu|_1\frac{2}{p_0}}\big]$ such that

$$\inf_{\lambda\in L^\nu_k}\frac{\Big(\sum\limits_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^p\Big)^{1/p}}{\Big(\sum\limits_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^{p_0}\Big)^{1/p_0}} \gtrsim \frac{2^{|\nu|_1(\frac{1}{p}-\frac{1}{p_0})}}{2}\,.$$

Here the constant behind $\gtrsim$ is the same as in (4.31). Consequently

$$\Big(\sum_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^{p_0}\Big)^{1/p_0} \lesssim 2^{-|\nu|_1(\frac{1}{p}-\frac{1}{p_0})}\Big(\sum_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^p\Big)^{1/p}. \tag{4.32}$$

holds for all $\lambda\in L^\nu_k$. We put

$$L^\mu=\bigoplus_{|\nu|_1=\mu}L^\nu_k\,.$$

Obviously $\dim L^\mu \asymp \big[\mu^{d-1}2^{\mu\frac{2}{p_0}}\big]$. Inserting (4.32) into (4.30) we have found

$$\frac{\|\lambda\ |(s^{0,\Omega}_{p,2}b)_\mu\|}{\|\lambda|(s^{t,\Omega}_{p_0,2}b)_\mu\|} \gtrsim \frac{2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\Big(\sum\limits_{|\nu|_1=\mu}\Big(\sum\limits_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^p\Big)^{2/p}\Big)^{1/2}}{\Big(\sum\limits_{|\nu|_1=\mu}2^{-2|\nu|_1(\frac{1}{p}-\frac{1}{p_0})}\Big(\sum\limits_{m\in A^\Omega_\nu}|\lambda_{\nu,m}|^p\Big)^{2/p}\Big)^{1/2}} = 2^{-t\mu}$$

for all $\lambda \in L^\mu$. In a view of (4.29) the desired result follows.
Step 2. We prove that

$$b_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big) \le x_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big) \tag{4.33}$$

for $1 < p_0 < \infty$. By p_0' we denote the conjugate number of p_0. From Lemma 4.7 and the duality of Kolmogorov and Gelfand numbers, see [84, Theorem 11.7.7], we deduce

$$\begin{aligned} b_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big) &= \big[c_{D_\mu-n+1}\big(id:\ (s^{0,\Omega}_{2,2}f)_\mu \to (s^{t,\Omega}_{p_0,2}f)_\mu\big)\big]^{-1} \\ &= \big[d_{D_\mu-n+1}\big(id:\ (s^{-t,\Omega}_{p_0',2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big)\big]^{-1}. \end{aligned} \tag{4.34}$$

Let $L_{D_\mu-n}$ be a subspace of $(s^{0,\Omega}_{2,2}f)_\mu$ with orthonormal basis $O^* = \{e^*_i,\ i = 1,\ldots,(D_\mu - n)\}$. By $O = \{e^j,\ j = 1,\ldots,n\}$ we denote an orthonormal system in $(s^{0,\Omega}_{2,2}f)_\mu$ such that $\{O^*, O\}$ is an orthonormal basis of $(s^{0,\Omega}_{2,2}f)_\mu$. Denote $(s^{0,\Omega}_{2,2}f)_{\mu,n}$ the span of O with the norm induced from $(s^{0,\Omega}_{2,2}f)_\mu$. From the definition of Kolmogorov numbers, see (4.2), we have

$$\begin{aligned} d_{D_\mu-n+1}\big(id:\ (s^{-t,\Omega}_{p_0',2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big) &= \inf_{L_{D_\mu-n}} \sup_{\|\lambda|(s^{-t,\Omega}_{p_0',2}f)_\mu\|=1} \inf_{\lambda_1 \in L_{D_\mu-n}} \|\lambda - \lambda_1|(s^{0,\Omega}_{2,2}f)_\mu\| \\ &= \inf_{L_{D_\mu-n}} \sup_{\|\lambda|(s^{-t,\Omega}_{p_0',2}f)_\mu\|=1} \Big\| \sum_{j=1}^n \langle\lambda, e^j\rangle e^j \Big| (s^{0,\Omega}_{2,2}f)_\mu \Big\| \\ &= \inf_{O} \sup_{\|\lambda|(s^{-t,\Omega}_{p_0',2}f)_\mu\|=1} \Big\| \sum_{j=1}^n \langle\lambda, e^j\rangle e^j \Big| (s^{0,\Omega}_{2,2}f)_\mu \Big\|. \end{aligned}$$

The infimum is taken over all orthonormal systems $O = \{e^j,\ j = 1,\ldots,n\}$. If we denote by Pr the projection from $(s^{-t,\Omega}_{p_0',2}f)_\mu$ onto $(s^{0,\Omega}_{2,2}f)_{\mu,n}$, then we get

$$d_{D_\mu-n+1}\big(id:\ (s^{-t,\Omega}_{p_0',2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big) = \inf_O \|\mathrm{Pr}:\ (s^{-t,\Omega}_{p_0',2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_{\mu,n}\|. \tag{4.35}$$

Property (c) yields

$$\begin{aligned} &x_n\big(J: (s^{0,\Omega}_{2,2}f)_{\mu,n} \to (s^{0,\Omega}_{2,2}f)_\mu\big) \\ &\qquad \le \|J_1:\ (s^{0,\Omega}_{2,2}f)_{\mu,n} \to (s^{t,\Omega}_{p_0,2}f)_\mu\| \cdot x_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big). \end{aligned}$$

Here J and J_1 are injections from respective spaces. Note that Pr is the adjoint operator of J_1. Hence we have

$$\begin{aligned} &x_n\big(J: (s^{0,\Omega}_{2,2}f)_{\mu,n} \to (s^{0,\Omega}_{2,2}f)_\mu\big) \\ &\qquad \le \|\mathrm{Pr}:\ (s^{-t,\Omega}_{p_0',2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_{\mu,n}\| \cdot x_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big). \end{aligned}$$

The equality

$$x_n\big(J: (s^{0,\Omega}_{2,2}f)_{\mu,n} \to (s^{0,\Omega}_{2,2}f)_\mu\big) = a_n\big(J: (s^{0,\Omega}_{2,2}f)_{\mu,n} \to (s^{0,\Omega}_{2,2}f)_\mu\big) = 1,$$

see property (e) in the definition of the s-numbers, implies

$$1 \ \le\ \|\mathrm{Pr}:\ (s^{-t,\Omega}_{p_0',2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_{\mu,n}\| \cdot x_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{2,2}f)_\mu\big).$$

This, in connection with (4.35), results in

$$\big(d_{D_\mu-n+1}\big(id:\ (s^{-t,\Omega}_{p_0',2}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big)\big)^{-1}\ \le\ x_n\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big)\,.$$

In view of (4.34) the inequality (4.33) follows.
Step 3. Proof of (ii). Let $p<2<p_0$. There exists some $\Theta\in(0,1)$ such that $\frac{1}{2}=\frac{1-\Theta}{p}+\frac{\Theta}{p_0}$ and consequently

$$\|\lambda|(s^{0,\Omega}_{2,2}f)_\mu\|\le\|\lambda|(s^{0,\Omega}_{p,2}f)_\mu\|^{1-\Theta}\cdot\|\lambda|(s^{0,\Omega}_{p_0,2}f)_\mu\|^{\Theta}$$

for all $\lambda\in(s^{0,\Omega}_{2,2}f)_\mu$. Now the interpolation property of the Weyl numbers, see Proposition 4.11, and property (a) of s-number yield

$$\begin{aligned}x_n\big(id:\ &(s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big)\\&\le x_n^{1-\Theta}\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p,2}f)_\mu\big)\cdot\|id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p_0,2}f)_\mu\|^{\Theta}\\&\le x_n^{1-\Theta}\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p,2}f)_\mu\big)\cdot 2^{-t\mu\Theta}\end{aligned}$$

for $n\in\mathbb{N}$. Finally, choosing $n=\big[\mu^{d-1}2^{\frac{2\mu}{p_0}}\big]$ and taking into account (4.33) and Step 1 the claim follows for Weyl numbers as well. The proof is complete. ■

Remark 4.33. The proof in Step 2 is similar to the proof of Satz 3.1 in [65].

Proposition 4.34. *Let $t\in\mathbb{R}$, $2\le p<p_0<\infty$ and $\frac{1}{r}=\frac{1/p-1/p_0}{1-2/p_0}$. Then we have*

$$\pi_{r,2}\big(id^*_\mu:(s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p,2}f)_\mu\big)\ \le\ 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}D_\mu^{\frac{1}{r}}.$$

Proof. We consider the case $2<p<p_0<\infty$. Let $\Theta=\frac{1/p-1/p_0}{1/2-1/p_0}$. Then we find

$$\frac{1}{p}=\frac{\Theta}{2}+\frac{1-\Theta}{p_0}\qquad\text{and}\qquad\frac{1}{r}=\frac{\Theta}{2}+\frac{1-\Theta}{\infty}.$$

By Hölder's inequality we obtain

$$\|\lambda|(s^{0,\Omega}_{p,2}f)_\mu\|\le\|\lambda|(s^{0,\Omega}_{2,2}f)_\mu\|^{\Theta}\cdot\|\lambda|(s^{0,\Omega}_{p_0,2}f)_\mu\|^{1-\Theta}$$

for all $\lambda\in(s^{0,\Omega}_{p,2}f)_\mu$. The definition of the absolutely $(r,2)$-summing norms yields

$$\begin{aligned}\pi_{r,2}\big(id^*_\mu:\ &(s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p,2}f)_\mu\big)\\&\le\pi_2^{\Theta}\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big)\cdot\|id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p_0,2}f)_\mu\|^{1-\Theta}\,.\end{aligned}$$

Note that the chain of embeddings

$$(s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{t,\Omega}_{p_0,p_0}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu$$

implies

$$\pi_2\big(id:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big)\le\pi_2\big(id:\ (s^{t,\Omega}_{p_0,p_0}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big),\tag{4.36}$$

since $\pi_{r,2}$ is an operator ideal, see [87, Theorem 1.2.3]. From this and Lemmas 4.24, 4.25 we derive

$$\begin{aligned}\pi_{r,2}\big(id^*_\mu:\ (s^{t,\Omega}_{p_0,2}f)_\mu\to(s^{0,\Omega}_{p,2}f)_\mu\big)&\lesssim\pi_2^{\Theta}\big(id:\ (s^{t,\Omega}_{p_0,p_0}f)_\mu\to(s^{0,\Omega}_{2,2}f)_\mu\big)\cdot 2^{-t\mu(1-\Theta)}\\&\lesssim\big[2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})}\pi_2\big(id:\ \ell^{D_\mu}_{p_0}\to\ell^{D_\mu}_2\big)\big]^{\Theta}\cdot 2^{-t\mu(1-\Theta)}\,.\end{aligned}$$

Finally, the equality $\pi_2(id:\ \ell^m_{p_0}\to\ell^m_2)=m^{\frac{1}{2}}$, see [84, page 309], yields the claimed estimate. The case $p=2$ is a consequence of (4.36). This finishes the proof. ■

The following corollary is a consequence of Lemma 4.14 and Proposition 4.34.

Corollary 4.35. *Let $2 \le p < p_0 < \infty$. Then*

$$x_n\big(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\Big(\frac{D_\mu}{n}\Big)^{1/r}, \qquad \frac{1}{r} = \frac{1/p - 1/p_0}{1-2/p_0}$$

holds for all $n \in \mathbb{N}$.

4.4 Weyl and Bernstein numbers of embeddings of sequence spaces

We need some further preparations.

Lemma 4.36. *Let $0 < p_0, p < \infty$ and $t > \big(\frac{1}{p_0} - \frac{1}{p}\big)_+$.*
(i) *If $p_0 \ge 2$, then we have*

$$\omega_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \lesssim \omega_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f)$$

(ii) *If $p_0 \le 2$, then we have*

$$\omega_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \lesssim \omega_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f)$$

Proof. We consider the continuous embeddings

$$s^{t,\Omega}_{p_0,2}f \to s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f\,, \quad p_0 \ge 2.$$

From property (c) of the s-numbers we obtain

$$\begin{aligned}\omega_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) &\le \omega_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f)\cdot \|id : s^{t,\Omega}_{p_0,2}f \to s^{t,\Omega}_{p_0,p_0}f\| \\ &\lesssim \omega_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f)\end{aligned}$$

This proves (i). By employing the chain of continuous embeddings

$$s^{t,\Omega}_{p_0,p_0}f \to s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f\,, \quad p_0 \le 2,$$

and a similar argument as above we obtain part (ii) as well. ■

Lemma 4.37. *Let $t, r \in \mathbb{R}$ and $0 < p, q, p_0, q_0 \le \infty$. Then*

$$\omega_n(id^1 : s^{t,\Omega}_{p_0,q_0}f \to s^{r,\Omega}_{p,q}f) \asymp \omega_n(id^2 : s^{t-r,\Omega}_{p_0,q_0}f \to s^{0,\Omega}_{p,q}f)\,, \qquad n \in \mathbb{N}\,.$$

Proof. We consider the commutative diagram

$$\begin{array}{ccc} s^{t,\Omega}_{p_0,q_0}f & \xrightarrow{id^1} & s^{r,\Omega}_{p,q}f \\ {\scriptstyle J_r}\big\downarrow & & \big\uparrow{\scriptstyle J_{-r}} \\ s^{t-r,\Omega}_{p_0,q_0}f & \xrightarrow{id^2} & s^{0,\Omega}_{p,q}f. \end{array}$$

Here J_r is the isomorphism defined in (4.10). Hence $\omega_n(id^1) \lesssim \omega_n(id^2)$. But

$$\begin{array}{ccc} s^{t-r,\Omega}_{p_0,q_0}f & \xrightarrow{id^2} & s^{0,\Omega}_{p,q}f \\ {\scriptstyle J_{-r}}\big\downarrow & & \big\uparrow{\scriptstyle J_r} \\ s^{t,\Omega}_{p_0,q_0}f & \xrightarrow{id^1} & s^{r,\Omega}_{p,q}f \end{array}$$

yields $\omega_n(id^2) \lesssim \omega_n(id^1)$ as well. The proof is complete. ■

4.4.1 The results for Weyl numbers

Now we are in position to investigate the asymptotic behaviour of the Weyl numbers of $id^*: s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f$. Here $q \in \{2, p_0\}$. Recall that if $q = p_0$ then the range of p_0 is $(0,\infty]$, otherwise $0 < p_0 < \infty$. We have to continue with the proof already started in (4.12)-(4.18). Always the positions of p_0 and p relative to 2 are of importance. Therefore we need to distinguish several cases.

The case $0 < p_0 \le 2 \le p < \infty$ and $t > \frac{1}{p_0} - \frac{1}{p}$

Theorem 4.38. *Let $0 < p_0 \le 2 \le p < \infty$ and $t > \frac{1}{p_0} - \frac{1}{p}$. Then*

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t+\frac{1}{2}-\frac{1}{p}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}\,, \qquad n \ge 2\,.$$

Proof. *Step 1.* Estimate from below. Since $p \ge 2$, from (4.19) and (4.21) with $\gamma = p$ we derive

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_n(id^{D_\mu}_{p_0,p})\,.$$

Next we choose $n = [\frac{D_\mu}{2}]$. Then from property (4.8c) we get

$$x_n(id^{D_\mu}_{p_0,p}) \gtrsim (D_\mu)^{\frac{1}{2}-\frac{1}{p_0}} \asymp (2^\mu\, \mu^{d-1})^{\frac{1}{2}-\frac{1}{p_0}}\,,$$

which implies

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})}\, \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\,.$$

Because of $2^\mu \asymp \frac{n}{\log^{d-1} n}$ we conclude

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim n^{-t+\frac{1}{2}-\frac{1}{p}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}$$

for $n \asymp \mu^{d-1}2^\mu$, $\mu \in \mathbb{N}_0$. By monotonicity of Weyl numbers, we extend this result to all $n \ge 2$.

Step 2. Estimate from above. For given J we choose $L > J$ large enough such that

$$2^{-L\left(t-(\frac{1}{p_0}-\frac{1}{p})_+\right)}\, L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})_+} = 2^{L(-t+\frac{1}{p_0}-\frac{1}{p})} \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{J(-t+\frac{1}{2}-\frac{1}{p})}\,. \tag{4.37}$$

For the sum in (4.18), we define

$$n_\mu := [D_\mu\, 2^{(J-\mu)\lambda}] \le \frac{D_\mu}{2}, \qquad J+1 \le \mu \le L\,, \tag{4.38}$$

where $\lambda > 1$ is at our disposal. We choose λ such that

$$t - \frac{1}{2} + \frac{1}{p} > \lambda\Big(\frac{1}{p_0} - \frac{1}{2}\Big) \tag{4.39}$$

which is always possible under the given restrictions. Then

$$\sum_{\mu=J+1}^{L} n_\mu \asymp J^{d-1}2^J \tag{4.40}$$

follows. If $p > 2$, we choose $\varepsilon > 0$ such that $2 \le p - \varepsilon$. From (4.22) we obtain

$$x_{n_\mu}(id^*_\mu) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_{n_\mu}(id^{D_\mu}_{p_0,p-\varepsilon}).$$

If $p = 2$, then (4.21) can be applied

$$x_{n_\mu}(id^*_\mu) \lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})}\, x_{n_\mu}(id^{D_\mu}_{p_0,2})\,.$$

Employing property (4.8c) we obtain

$$\begin{aligned} x_{n_\mu}(id^*_\mu) &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, \big(\mu^{d-1}\, 2^{\mu}\, 2^{(J-\mu)\lambda}\big)^{\frac{1}{2}-\frac{1}{p_0}} \\ &= \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})}\, 2^{(J-\mu)\lambda(\frac{1}{2}-\frac{1}{p_0})}\,. \end{aligned}$$

Our special choice of λ in (4.39) yields

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{J(-t+\frac{1}{2}-\frac{1}{p})}\,. \tag{4.41}$$

Here $\rho = 1$ since $2 \le p < \infty$, see (4.14). Inserting (4.37) and (4.41) into (4.18) leads to

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{J(-t+\frac{1}{2}-\frac{1}{p})}\,,$$

where n depends on J. More exactly,

$$n = n_J = 1 + \sum_{\mu=0}^{L}(n_\mu - 1) \asymp J^{d-1}\, 2^J \le B J^{d-1} 2^J\,,$$

see (4.16) and (4.40). There B is independent of J. Without loss of generality we assume $B \in \mathbb{N}$. Then we conclude from the monotonicity of the Weyl numbers

$$\begin{aligned} x_{BJ^{d-1}2^J}(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) &\lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{J(-t+\frac{1}{2}-\frac{1}{p})} \\ &\asymp (BJ^{d-1}2^J)^{-t+\frac{1}{2}-\frac{1}{p}} (\log(BJ^{d-1}2^J))^{(d-1)(t+\frac{1}{p}-\frac{1}{p_0})}\,. \end{aligned}$$

Hence, our proof works for certain subsequence $\{BJ^{d-1}2^J\}_{j=1}^\infty$ of the natural numbers. Employing one more time the monotonicity of the Weyl numbers and in addition its polynomial behaviour we can switch from the subsequence $\{BJ^{d-1}2^J\}_{j=1}^\infty$ to $n \in \mathbb{N}$ in this formula by possibly changing the constant behind $\lesssim$. This finishes our proof. ■

Theorem 4.39. *Let* $0 < p_0 \le 2 \le p < \infty$ *and* $t > \frac{1}{p_0} - \frac{1}{p}$. *Then*

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t+\frac{1}{2}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{2}+\frac{1}{p})}\,, \qquad n \ge 2.$$

Proof. *Step 1.* Estimate from below. Because of $p_0 \le 2 \le p$, Lemma 4.28 and (4.23) imply

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})} x_n(id^{D_\mu}_{2,p}).$$

We choose $n = \big[D_\mu/2\big]$. Then (4.8a) yields $x_n(id^{D_\mu}_{2,p}) \asymp 1$. Hence

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{p}-\frac{1}{2})} \gtrsim n^{-t+\frac{1}{2}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{2}+\frac{1}{p})}.$$

Step 2. Estimate from above. We choose $L > J$ such that

$$2^{L(-t+\frac{1}{p_0}-\frac{1}{p})} \lesssim 2^{J(-t+\frac{1}{2}-\frac{1}{p})}. \tag{4.42}$$

Next we define n_μ and λ as (4.38) and (4.39). Hence (4.40) follows. Now (4.25) and (4.8b) yield

$$\begin{aligned} x_{n_\mu}(id^*_\mu) &\lesssim \mu^{(d-1)(\frac{1}{p_0}-\frac{1}{2})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_{n_\mu}(id^{D_\mu}_{p_0,2}) \\ &\lesssim \mu^{(d-1)(\frac{1}{p_0}-\frac{1}{2})} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} (D_\mu 2^{(J-\mu)\lambda})^{\frac{1}{2}-\frac{1}{p_0}} \\ &\asymp 2^{\mu(-t+\frac{1}{2}-\frac{1}{p})} 2^{(J-\mu)\lambda(\frac{1}{2}-\frac{1}{p_0})}. \end{aligned}$$

Taking into account the condition (4.39), we obtain (here $\rho = 1$ since $2 \le p < \infty$)

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) \lesssim 2^{J(-t+\frac{1}{2}-\frac{1}{p})}.$$

see (4.18). From this together with (4.42) we have found

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \lesssim 2^{J(-t+\frac{1}{2}-\frac{1}{p})}.$$

By arguing as at the end of the proof of Theorem 4.38 we finish the proof. ■

The case $2 < p_0 \le p < \infty$, $t > \frac{1}{p_0} - \frac{1}{p}$

Theorem 4.40. *Let $2 < p_0 \le p < \infty$, $q \in \{p_0, 2\}$ and $t > \frac{1}{p_0} - \frac{1}{p}$. Then we have*

$$x_n(id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t+\frac{1}{p_0}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}, \qquad n \ge 2.$$

Proof. Because of $2 < p_0$, it is enough to estimate from below for $q = 2$ and from above for $q = p_0$, see Lemma 4.36.
Step 1. Estimate from below with $q = 2$. Since $2 < p_0$ we choose $\varepsilon > 0$ such that $2 \le p_0 - \varepsilon$. Then Lemma 4.28 and (4.24) with $p \ge 2$ yield

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_n(id^{D_\mu}_{p_0-\varepsilon,p}).$$

Now property (4.8a) with $n = \left[\frac{D_\mu}{2}\right]$ imply $x_n(id^{D_\mu}_{p_0-\varepsilon,p}) \asymp 1$, which results in

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} \gtrsim n^{-t+\frac{1}{p_0}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}.$$

Step 2. Estimate from above with $q = p_0$. First we assume that $p_0 < p$. Let $\{e^{\nu,m},\ \nu \in \mathbb{N}^d_0,\ m \in A^\Omega_\nu\}$ be the canonical orthonormal basis of $s^{0,\Omega}_{2,2}f$. For $J \in \mathbb{N}$ and $\lambda \in s^{t,\Omega}_{p_0,p_0}f$ we put

$$S_J\lambda := \sum_{\mu=0}^{J} \sum_{|\nu|_1=\mu} \sum_{m\in A^\Omega_\nu} \lambda_{\nu,m} e^{\nu,m}. \tag{4.43}$$

Obviously, we have

$$\operatorname{rank}(S_J) \asymp \sum_{\mu=0}^{J} 2^\mu \mu^{d-1} \asymp 2^J J^{d-1} \le B 2^J J^{d-1}$$

for some $B \in \mathbb{N}$ independent of J. As a consequence of Corollary 4.26 we obtain

$$\|id^* - S_J : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f\| \le \sum_{\mu=J+1}^{\infty} \|id_\mu : (s^{t,\Omega}_{p_0,p_0} f)_\mu \to (s^{0,\Omega}_{p,2} f)_\mu\| \le 2^{-J\left(t-\frac{1}{p_0}+\frac{1}{p}\right)}.$$

This implies

$$a_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \lesssim 2^{-J\left(t-\frac{1}{p_0}+\frac{1}{p}\right)}.$$

From the inequality $x_n \le a_n$ and the monotonicity of Weyl numbers we obtain the desired estimate. For the case $p_0 = p$ we argue as in the proof of Theorem 4.41 below. The proof is complete. ■

The case $\max(2,p) < p_0 \le \infty$**,** $t > \frac{1/\max(p,2)-1/p_0}{p_0/2-1}$

Theorem 4.41. *Let* $2 \le p < p_0 \le \infty$ *and* $t > \frac{1/p-1/p_0}{p_0/2-1}$. *Then we have*

$$x_n(id^* : s^{t,\Omega}_{p_0,q} f \to s^{0,\Omega}_{p,2} f) \asymp n^{-t+\frac{1}{p_0}-\frac{1}{p}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{p})}, \qquad n \ge 2.$$

Proof. As a consequence of Lemma 4.36 we estimate from below for $q = 2$ and from above for $q = p_0$.
Step 1. Estimate from below with $q = 2$. We follow the arguments used in Step 1 in the proof of Theorem 4.40. Note that instead of using property (4.8a) we employ Lemma 4.16 (b-ii).
Step 2. Estimate from above with $q = p_0$. For given J we can choose L large enough such that

$$2^{-L(t-(\frac{1}{p_0}-\frac{1}{p})_+)} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})_+} = 2^{-Lt} L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} \le 2^{J(-t+\frac{1}{p_0}-\frac{1}{p})}. \tag{4.44}$$

We define n_μ for $J+1 \le \mu \le L$ as (4.38). Hence (4.40) follows. The restriction $t > \frac{1/p-1/p_0}{p_0/2-1}$ implies

$$-t + \frac{1}{p_0} - \frac{1}{p} + \frac{1/p - 1/p_0}{1 - 2/p_0} < 0.$$

If $p > 2$ we choose $\varepsilon > 0$ such that $2 \le p - \varepsilon$ and

$$-t + \frac{1}{p_0} - \frac{1}{p} + \frac{1/(p-\varepsilon) - 1/p_0}{1 - 2/p_0} < 0. \tag{4.45}$$

In this situation we derive from Lemma 4.16 (b-i)

$$x_{n_\mu}(id^{D_\mu}_{p_0,p-\varepsilon}) \lesssim \Big(\frac{D_\mu}{n_\mu}\Big)^{1/r} \asymp 2^{-\frac{(J-\mu)\lambda}{r}}, \qquad \frac{1}{r} := \frac{1/(p-\varepsilon) - 1/p_0}{1 - 2/p_0}.$$

The estimate (4.22) guarantees

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) \lesssim \sum_{\mu=J+1}^{L} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} x_{n_\mu}(id^{D_\mu}_{p_0,p-\varepsilon}). \tag{4.46}$$

Here $\rho = 1$, see (4.14). In case $p = 2$, again Lemma 4.16 (b-i) yields

$$x_{n_\mu}(id^{D_\mu}_{p_0,2}) \lesssim \Big(\frac{D_\mu}{n_\mu}\Big)^{1/2} \asymp 2^{-\frac{(J-\mu)\lambda}{r}}, \qquad \frac{1}{r} := \frac{1}{2}.$$

From (4.21) we obtain

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) \lesssim \sum_{\mu=J+1}^{L} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})}\, x_{n_\mu}(id^{D_\mu}_{p_0,2}). \tag{4.47}$$

Now (4.46) and (4.47) yield

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) \lesssim \sum_{\mu=J+1}^{L} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, 2^{-\frac{(J-\mu)\lambda}{r}} = \sum_{\mu=J+1}^{L} 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{\lambda}{r})}\, 2^{-\frac{J\lambda}{r}}.$$

The condition (4.45) can be rewritten as $-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}<0$. This allows us to choose $\lambda > 1$ such that $-t+\frac{1}{p_0}-\frac{1}{p}+\frac{\lambda}{r}<0$. Then

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id^*_\mu) \lesssim 2^{J(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{\lambda}{r})}\, 2^{-\frac{J\lambda}{r}} = 2^{J(-t+\frac{1}{p_0}-\frac{1}{p})}$$

follows. Inserting this and (4.44) into (4.18) we find

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \lesssim 2^{J(-t+\frac{1}{p_0}-\frac{1}{p})}$$

and this is enough to prove the estimate from above; compare with the end of the proof of Theorem 4.38. ■

Theorem 4.42. *Let* $0 < p \le 2 < p_0 \le \infty$ *and* $t > \frac{1}{p_0}$. *Then*

$$x_n(id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t+\frac{1}{p_0}-\frac{1}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \quad n \ge 2.$$

Proof. *Step 1.* Estimate from below. Because $2 < p_0$ we choose $\varepsilon > 0$ such that $2 < p_0 - \varepsilon$. Then Lemma 4.28 and (4.24) yield

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}x_n(id^{D_\mu}_{p_0-\varepsilon,p})$$

Employing Lemma 4.16 (c-i) with $n = \big[\frac{D_\mu}{2}\big]$ we have found

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}D_\mu^{\frac{1}{p}-\frac{1}{2}} \asymp 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})}.$$

This implies the lower estimate for the case $q = 2$.
Step 2. Estimate from above. Since $p \le 2 < p_0$ we have continuous embeddings

$$s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{2,2}f \to s^{0,\Omega}_{p,2}f.$$

Property (c) of $s-$numbers and Theorem 4.41 result in

$$\begin{aligned} x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) &\lesssim x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{2,2}f) \\ &\lesssim n^{-t+\frac{1}{p_0}-\frac{1}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \qquad n \ge 2. \end{aligned}$$

In view of Lemma 4.36 we finish the proof. ■

The case $\max(p,2) < p_0 < \infty$**,** $0 < t < \frac{1/\max(p,2)-1/p_0}{p_0/2-1}$

Proposition 4.43. *Let* $0 < p \le 2 < p_0 < \infty$ *and* $t > 0$. *Then we have*
(i)
$$x_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \gtrsim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} ,$$
(ii) *and*
$$x_n(id^* : s^{t,\Omega}_{p_0,2} f \to s^{0,\Omega}_{p,2} f) \gtrsim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)\frac{tp_0}{2}} , \quad n \ge 2.$$

Proof. *Step 1.* Proof of (i). Since $p \le 2$, combining (4.19) and (4.20) we obtain
$$x_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_n(id^{D_\mu}_{p_0,p}) .$$
With $n := [D_\mu^{\frac{2}{p_0}}]$ Lemma 4.16 (c-ii) yields
$$x_n(id^{D_\mu}_{p_0,p}) \gtrsim D_\mu^{\frac{1}{p}-\frac{1}{p_0}} \gtrsim (\mu^{d-1}\, 2^\mu)^{\frac{1}{p}-\frac{1}{p_0}} .$$
Hence
$$x_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{-t\mu} \gtrsim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} .$$
Again the monotonicity of Weyl numbers implies the estimate for all $n \ge 2$.
Step 2. Proof of (ii). From Lemmas 4.28 and 4.32 we have
$$x_n(id^* : s^{t,\Omega}_{p_0,2} f \to s^{0,\Omega}_{p,2} f) \gtrsim 2^{-t\mu}, \quad n = \left[\mu^{(d-1)} 2^{\frac{2\mu}{p_0}}\right] .$$
Rewriting this in dependence of n we get desired estimate. ■

Proposition 4.44. *Let* $2 \le p < p_0 < \infty$ *and* $0 < t < \frac{1/p-1/p_0}{p_0/2-1}$. *Then we have*
(i)
$$x_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \lesssim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} ,$$
(ii) *and*
$$x_n(id^* : s^{t,\Omega}_{p_0,2} f \to s^{0,\Omega}_{p,2} f) \lesssim n^{-\frac{tp_0}{2}} (\log n)^{(d-1)\frac{tp_0}{2}} , \quad n \ge 2.$$

Proof. *Step 1.* Proof of (i). For fixed $J \in \mathbb{N}$ we choose
$$L := \Big[\frac{p_0}{2} J + (d-1)(\frac{p_0}{2} - 1) \log J\Big] .$$
This results in the estimate
$$\begin{aligned} 2^{-L(t-(\frac{1}{p_0}-\frac{1}{p})_+)}\, L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})_+} &= 2^{-tL}\, L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})} \\ &\lesssim 2^{-\frac{p_0}{2}Jt}\, J^{(d-1)(t-\frac{tp_0}{2}+\frac{1}{2}-\frac{1}{p_0})} . \end{aligned} \tag{4.48}$$
We define
$$n_\mu := \left[D_\mu\, 2^{\{(\mu-L)\beta+J-\mu\}}\right] \le D_\mu , \qquad J+1 \le \mu \le L , \tag{4.49}$$
where $\beta > 0$ will be fixed later on. Consequently (4.40) follows. Employing Lemma 4.16 (b-i) we get
$$x_{n_\mu}(id^{D_\mu}_{p_0,p}) \lesssim \Big(\frac{D_\mu}{n_\mu}\Big)^{1/r} \lesssim 2^{-\frac{(\mu-L)\beta+J-\mu}{r}} , \qquad \frac{1}{r} := \frac{1/p-1/p_0}{1-2/p_0} .$$

We continue by applying (4.21)

$$\begin{aligned}\sum_{\mu=J+1}^{L} x_{n_\mu}\big(id_\mu^* : (s^{t,\Omega}_{p_0,p_0}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) &\lesssim \sum_{\mu=J+1}^{L} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_{n_\mu}(id^{D_\mu}_{p_0,p}) \\ &\lesssim \sum_{\mu=J+1}^{L} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, 2^{-\frac{(\mu-L)\beta+J-\mu}{r}} \\ &= \sum_{\mu=J+1}^{L} \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}-\frac{\beta}{r})}\, 2^{\frac{L\beta-J}{r}}\,.\end{aligned}$$

Note that here $\rho = 1$, see (4.14). The condition $t < \frac{1/p-1/p_0}{p_0/2-1}$ implies $-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r} > 0$. Because of this, we can choose $\beta > 0$ such that

$$-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}-\frac{\beta}{r} > 0.$$

Then

$$\begin{aligned}\sum_{\mu=J+1}^{L} x_{n_\mu}\big(id_\mu^* : (s^{t,\Omega}_{p_0,p_0}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\big) &\lesssim L^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{L(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}-\frac{\beta}{r})}\, 2^{\frac{L\beta-J}{r}} \\ &= L^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{L(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r})}\, 2^{-\frac{J}{r}}\end{aligned} \tag{4.50}$$

follows. Inserting the definition of L we conclude

$$L^{(d-1)(\frac{1}{2}-\frac{1}{p})} \lesssim J^{(d-1)(\frac{1}{2}-\frac{1}{p})}$$

and

$$\begin{aligned}2^{L(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r})}\, 2^{-\frac{J}{r}} &\lesssim 2^{\left[\frac{p_0}{2}J+(d-1)(\frac{p_0}{2}-1)\log J\right]\left[-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r}\right]} 2^{-\frac{J}{r}} \\ &\lesssim 2^{-\frac{tp_0}{2}J}\, J^{(d-1)(\frac{p_0}{2}-1)(-t+\frac{1}{p_0}-\frac{1}{p}+\frac{1}{r})} \\ &= 2^{-\frac{tp_0}{2}J}\, J^{(d-1)(t-\frac{tp_0}{2}-\frac{1}{p_0}+\frac{1}{p})}\,.\end{aligned}$$

Now (4.50) yields

$$\sum_{\mu=J+1}^{L} x_{n_\mu}(id_\mu^*) \lesssim J^{(d-1)(t-\frac{tp_0}{2}-\frac{1}{p_0}+\frac{1}{2})}\, 2^{-\frac{tp_0}{2}J}\,.$$

This, together with (4.48), has to be inserted into (4.18)

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \lesssim J^{(d-1)(t-\frac{tp_0}{2}-\frac{1}{p_0}+\frac{1}{2})}\, 2^{-\frac{tp_0}{2}J}$$

which implies the estimate of part (i).
Step 2. Proof of (ii). We choose $L = \left[J\frac{p_0}{2}\right]$ and define n_μ for $J+1 \le \mu \le L$ as (4.49). Employing Corollary 4.35 we obtain

$$\begin{aligned}x_{n_\mu}(id_\mu^* : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu) &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\Big(\frac{D_\mu}{n_\mu}\Big)^{1/r} \\ &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})} 2^{-\frac{(\mu-L)\beta+J-\mu}{r}}.\end{aligned}$$

Following the same argument as in Step 1 we find

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \lesssim 2^{-\frac{tp_0}{2}J}.$$

Finally, we finish the proof by the standard monotonicity argument. ■

Theorem 4.45. *Let* $\max(p,2) < p_0 < \infty$ *and* $0 < t < \frac{1/\max(p,2)-1/p_0}{p_0/2-1}$. *Then we have*
(i)

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-\frac{tp_0}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})},$$

(ii) *and*

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-\frac{tp_0}{2}}(\log n)^{(d-1)\frac{tp_0}{2}}, \quad n \geq 2.$$

Proof. *Step 1.* The case $2 \leq p$. We employ the chain of embeddings

$$s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f \to s^{0,\Omega}_{2,2}f$$

and property (c) of the s-numbers to obtain

$$x_n(id : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{2,2}f) \lesssim x_n(id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f).$$

Now the results in Propositions 4.43 and 4.44 imply the estimate in this case.
Step 2. The case $p \leq 2$. By considering the chain of embeddings

$$s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{2,2}f \to s^{0,\Omega}_{p,2}f$$

and using the same argument as in Step 1 we obtain the claimed estimate in this case as well. ■

The case $0 < p_0, p \leq 2$

Theorem 4.46. *Let* $0 < p_0, p \leq 2$, $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$ *and* $t \neq \frac{1}{p_0} - \frac{1}{2}$. *Then we have*

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t}(\log n)^{(d-1)(t+\frac{1}{2}-\frac{1}{p_0})_+}, \quad n \geq 2\,.$$

Proof. *Step 1.* Estimate from below.
Substep 1.1. The case $t < \frac{1}{p_0} - \frac{1}{2}$. Lemma 4.28 and (4.8d) with $n = 2^\mu$ yield

$$x_n(I_\mu) \geq 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_n(id^{A_\mu}_{p_0,p}) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, 2^{\mu(\frac{1}{p}-\frac{1}{p_0})} = 2^{-\mu t} \asymp n^{-t}\,,$$

which implies

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim n^{-t}.$$

Substep 1.2. The case $t > \frac{1}{p_0} - \frac{1}{2}$. From (4.19) and (4.20) we have

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}\, 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_n(id^{D_\mu}_{p_0,p})\,.$$

We choose $n := [D_\mu/2]$. Then property (4.8b) (see also (4.8d)) leads to

$$x_n(id^{D_\mu}_{p_0,p}) \gtrsim D_\mu^{\frac{1}{p}-\frac{1}{p_0}} \gtrsim (\mu^{d-1}2^\mu)^{\frac{1}{p}-\frac{1}{p_0}}\,,$$

which implies

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}\, 2^{-t\mu}\,.$$

This proves the lemma if $t + \frac{1}{2} - \frac{1}{p_0} > 0$..

Step 2. Estimate from above.

Substep 2.1. The case $t > \frac{1}{p_0} - \frac{1}{2}$. Using chain of continuous embeddings

$$s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{2,2} f \to s^{0,\Omega}_{p,2} f\,.$$

together with the results of Theorem 4.38 and property (c) of the s-numbers we obtain the upper estimate in this case.

Step 2.2. The case $0 < p \le p_0 < 2$ and $0 < t < \frac{1}{p_0} - \frac{1}{2}$. For given $J \in \mathbb{N}$ we choose $L := J + (d-1)\,[\log J]$. Here we assume $d > 1$. If $d = 1$ we come back to the isotropic situation in one dimension, see above-mentioned references. With this L we have

$$2^{-L(t-(\frac{1}{p_0}-\frac{1}{p})_+)}\, L^{(d-1)(\frac{1}{2}-\frac{1}{p_0})_+} = 2^{-Lt} \asymp 2^{-tJ}\, J^{(d-1)(-t)}\,. \tag{4.51}$$

We define n_μ for $J+1 \le \mu \le L$, as (4.49). Then (4.40) follows. Property (4.8c) yields

$$x_{n_\mu}(id^{D_\mu}_{p_0,2}) \lesssim \big(D_\mu 2^{(\mu-L)\beta+J-\mu}\big)^{\frac{1}{2}-\frac{1}{p_0}}\,.$$

This, in connection with (4.20) ($\delta = 2$), leads to

$$\begin{aligned}\sum_{\mu=J+1}^{L} x^\rho_{n_\mu}(id^*_\mu) &\lesssim \sum_{\mu=J+1}^{L} 2^{\mu\rho(-t+\frac{1}{p_0}-\frac{1}{2})}\big(D_\mu\, 2^{(\mu-L)\beta+J-\mu}\big)^{\rho(\frac{1}{2}-\frac{1}{p_0})}\\ &\lesssim \sum_{\mu=J+1}^{L} 2^{\mu\rho\big(-t+\frac{1}{p_0}-\frac{1}{2}+(\frac{1}{2}-\frac{1}{p_0})\beta\big)}\,\big(\mu^{(d-1)}2^{-L\beta+J}\big)^{\rho(\frac{1}{2}-\frac{1}{p_0})}\,.\end{aligned}$$

Because of $t < \frac{1}{p_0} - \frac{1}{2}$ we can select $\beta > 0$ such that

$$-t + \frac{1}{p_0} - \frac{1}{2} + (\frac{1}{2} - \frac{1}{p_0})\beta > 0\,.$$

Consequently

$$\begin{aligned}\sum_{\mu=J+1}^{L} x^\rho_{n_\mu}(id^*_\mu) &\lesssim 2^{L\rho\big(-t+\frac{1}{p_0}-\frac{1}{2}+(\frac{1}{2}-\frac{1}{p_0})\beta\big)}\,\big(L^{(d-1)}\,2^{-L\beta+J}\big)^{\rho(\frac{1}{2}-\frac{1}{p_0})}\\ &= 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{2})}\,\big(L^{(d-1)}\,2^{J}\big)^{\rho(\frac{1}{2}-\frac{1}{p_0})}\\ &\lesssim 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{2})}\,\big(J^{(d-1)}\,2^{J}\big)^{\rho(\frac{1}{2}-\frac{1}{p_0})}\\ &= 2^{L\rho(-t)}\,2^{L(\frac{1}{p_0}-\frac{1}{2})}\,\big(J^{(d-1)}\,2^{J}\,\big)^{\rho(\frac{1}{2}-\frac{1}{p_0})}\,.\end{aligned} \tag{4.52}$$

Observe

$$2^{L(\frac{1}{p_0}-\frac{1}{2})}\,\big(J^{(d-1)}\,2^J\big)^{\frac{1}{2}-\frac{1}{p_0}} = 2^{(J+(d-1)[\log J])(\frac{1}{p_0}-\frac{1}{2})}\big(J^{(d-1)}\,2^J\big)^{\frac{1}{2}-\frac{1}{p_0}} \asymp 1\,.$$

Replacing L by $J + (d-1)[\log J]$ in (4.52) we obtain

$$\sum_{\mu=J+1}^{L} x^{\rho}_{n_\mu}(id^*_\mu) \lesssim 2^{-L\rho t} \lesssim (2^J\, J^{d-1})^{-\rho t}.$$

This inequality, together with (4.51), yield

$$x_{n_J}(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \lesssim n_J^{-t}\,,$$

where $n_J = BJ^{d-1}2^J$ for some $B \in \mathbb{N}$. Now we can continue as at the end of the proof of Theorem 4.38.

Step 2.3. The case $0 < p_0 < p < 2$ and $\frac{1}{p_0} - \frac{1}{p} < t < \frac{1}{p_0} - \frac{1}{2}$. We split the sum in (4.18) into two terms

$$x^{\rho}_n(id^*) \lesssim \sum_{\mu=J+1}^{K} x^{\rho}_{n_\mu}(id^*_\mu) + \sum_{\mu=K+1}^{L} x^{\rho}_{n_\mu}(id^*_\mu) + 2^{L\rho(-t+\frac{1}{p_0}-\frac{1}{p})}\,. \tag{4.53}$$

We define $K := J + (d-1)[\log J]$ (as above we assume $d > 1$) and

$$n_\mu := \begin{cases} \big[D_\mu\, 2^{(\mu-K)\beta+J-\mu}\big] & \text{if} \quad J+1 \le \mu \le K\,, \\ \big[J^{d-1} 2^J\, 2^{(K-\mu)\gamma}\big] & \text{if} \quad K+1 \le \mu \le L\,. \end{cases}$$

Here β, $\gamma > 0$ will be fixed later. The condition β, $\gamma > 0$ implies (4.40). Making use of the same arguments as in Substep 2.2 we find

$$\sum_{\mu=J+1}^{K} x^{\rho}_{n_\mu}(id^*_\mu) \lesssim 2^{-L\rho t} \lesssim 2^{-tJ\rho}\, J^{-(d-1)\rho t}\,. \tag{4.54}$$

Now we estimate the second sum in (4.53). Therefore we consider the following splitting of n_μ, $K+1 \le \mu \le L$

$$n_\mu \asymp J^{d-1}\, 2^J\, 2^{(K-\mu)\gamma} = J^{d-1}\, 2^\mu\, 2^{K-\mu}\, 2^{-(d-1)[\log J]}\, 2^{(K-\mu)\gamma} \asymp 2^\mu\, 2^{(K-\mu)(\gamma+1)}\,,$$

where we used the definition of K. Observe $n_\mu \le D_\mu/2$. The inequality (4.20) with $\delta = p$ and property (4.8b) lead to the estimate

$$\begin{aligned} x_{n_\mu}(id^*_\mu) &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, x_n(id^{D_\mu}_{p_0,p}) \\ &\lesssim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{p})}\, \big(2^\mu\, 2^{(K-\mu)(\gamma+1)}\big)^{\frac{1}{p}-\frac{1}{p_0}} = 2^{-\mu t}\, 2^{(K-\mu)(\gamma+1)(\frac{1}{p}-\frac{1}{p_0})}\,. \end{aligned}$$

This implies

$$\sum_{\mu=K+1}^{L} x^{\rho}_{n_\mu}(id^*_\mu) \lesssim \sum_{\mu=K+1}^{L} 2^{-\mu\rho t}\, 2^{(K-\mu)(\gamma+1)(\frac{1}{p}-\frac{1}{p_0})\rho}\,.$$

Choosing $\gamma > 0$ such that $t > (\gamma+1)\left(\frac{1}{p_0} - \frac{1}{p}\right)$ we conclude

$$\sum_{\mu=K+1}^{L} x^{\rho}_{n_\mu}(id^*_\mu) \lesssim 2^{-Lt\rho} \asymp 2^{-tJ\rho} J^{-(d-1)t\rho}\,.$$

Hence, inserting the previous inequality and (4.54) into (4.53) and choosing L large enough

$$x_n(id^* : s^{t,\Omega}_{p_0,p_0} f \to s^{0,\Omega}_{p,2} f) \lesssim 2^{-tJ}\, J^{(d-1)(-t)}$$

follows. Based on this estimate and (4.40) one can finish the proof as before. ■

Theorem 4.47. *Let $0 < p, p_0 \le 2$ and $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$. Then we have*

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t}(\log n)^{(d-1)t}\,, \qquad n \ge 2.$$

Proof. *Step 1.* Estimate from below. Since $p_0, p \le 2$, from Lemma 4.28 and (4.23) we have

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}2^{\mu(-t+\frac{1}{2}-\frac{1}{p})}x_n(id^{D_\mu}_{2,p}).$$

By choosing $n = \left[D_\mu/2\right]$ together with (4.8d) we obtain

$$\begin{aligned}x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) &\gtrsim \mu^{(d-1)(\frac{1}{2}-\frac{1}{p})}2^{\mu(-t+\frac{1}{2}-\frac{1}{p})}(D_\mu)^{\frac{1}{p}-\frac{1}{2}}\\ &\asymp 2^{\mu(-t)}\\ &\asymp n^{-t}(\log n)^{(d-1)t}.\end{aligned}$$

Step 2. Estimate from above.
Substep 2.1. The case $0 < p \le p_0 \le 2$ and $t > 0$. We define the operator S_J, $J \in \mathbb{N}$, as (4.43). In a view of Corollary 4.26 we have found

$$\|id^* - S_J : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f\| \le \sum_{\mu=J+1}^{\infty} \|id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu\| \le 2^{-Jt}.$$

This leads to

$$x_{B2^JJ^{d-1}}(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \le a_{B2^JJ^{d-1}}(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \lesssim 2^{-Jt}\,,$$

for some $B \in \mathbb{N}$ independent of J. Now the monotonicity of the Weyl numbers implies

$$x_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \lesssim n^{-t}(\log n)^{(d-1)t}.$$

Substep 2.2. The case $0 < p_0 < p < 2$ and $t > \frac{1}{p_0} - \frac{1}{p}$. By putting $\Theta = \frac{1/p_0 - 1/p}{1/p_0 - 1/2} \in (0,1)$, $t_1 = \frac{1}{p_0} - \frac{1}{p}$ and $t_2 = \frac{1}{2} - \frac{1}{p}$ we obtain from the condition $t > \frac{1}{p_0} - \frac{1}{p}$ that

$$(1-\Theta)t_1 + \Theta t_2 = 0, \quad \frac{1}{p} = \frac{1-\Theta}{p_0} + \frac{\Theta}{2}, \quad t - t_1 > 0 \quad \text{and} \quad t - t_2 > \frac{1}{p_0} - \frac{1}{2}$$

hold. Then complex interpolation yields

$$[s^{t_1,\Omega}_{p_0,2}f, s^{t_2,\Omega}_{2,2}f]_\Theta = s^{0,\Omega}_{p,2}f\,,$$

see Proposition 1.49 or [140, Theorem 4.6]. Employing the lifting property, see Lemma 4.37, results in Step 2 and Theorem 4.39 we conclude that

$$\begin{aligned}x_n(id : s^{t,\Omega}_{p_0,2}f \to s^{t_1,\Omega}_{p_0,2}f) &\asymp x_n(id : s^{t-t_1,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p_0,2}f)\\ &\asymp n^{-t+t_1}(\log n)^{(d-1)(t-t_1)}\end{aligned} \tag{4.55}$$

and

$$\begin{aligned}x_n(id : s^{t,\Omega}_{p_0,2}f \to s^{t_2,\Omega}_{2,2}f) &\asymp x_n(id : s^{t-t_2,\Omega}_{p_0,2}f \to s^{0,\Omega}_{2,2}f)\\ &\asymp n^{-t+t_2}(\log n)^{(d-1)(t-t_2)}.\end{aligned} \tag{4.56}$$

The interpolation property of the Weyl numbers, see Proposition 4.11, results in

$$x_{2n-1}(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \lesssim x_n^{1-\Theta}(id : s^{t,\Omega}_{p_0,2}f \to s^{t_1,\Omega}_{p_0,2}f) \cdot x_n^{\Theta}(id : s^{t,\Omega}_{p_0,2}f \to s^{t_2,\Omega}_{2,2}f).$$

Inserting (4.55) and (4.56) into this inequality we complete the proof. ■

4.4.2 The results for Bernstein numbers

Let us first recall the behaviour of entropy numbers of the embeddings $id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f$, see [140, Theorem 4.11].

Proposition 4.48. (i) *Let* $0 < p_0, p < \infty$ *and* $t > (\frac{1}{p_0} - \frac{1}{p})_+$. *Then we have*

$$e_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t}(\log n)^{(d-1)t}, \qquad n \geq 2.$$

(ii) *Let* $0 < p_0 \leq \infty$, $0 < p < \infty$ *and* $t > \max\big(0, \frac{1}{p_0} - \frac{1}{2}, \frac{1}{p_0} - \frac{1}{p}\big)$. *Then we have*

$$e_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \qquad n \geq 2\,.$$

Remark 4.49. Further estimates of the decay of entropy numbers related to embeddings $id : S^t_{p_0,q}A(\Omega) \to L_p(\Omega)$ (and also $id : S^t_{p_0,q_0}A(\Omega) \to S^0_{p,q}A(\Omega)$) can be found in Belinsky [10], D. Dũng [28], Temlyakov [117] and Vybiral [140].

Now we are in position to formulate the results for Bernstein numbers.

Theorem 4.50. *Let* $1 < p_0, p < \infty$ *and* $t > (\frac{1}{p_0} - \frac{1}{p})_+$. *Then we have*

$$b_n(id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-\beta}(\log n)^{(d-1)\beta}, \quad n \geq 2\,,$$

where

(i) $\beta = t$ *if* $p_0 \leq p$ *or* $p \leq p_0 \leq 2$;

(ii) $\beta = t - \frac{1}{p_0} + \frac{1}{\max(p,2)}$ *if* $\max(p,2) < p_0,\ t > \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$;

(iii) $\beta = \frac{tp_0}{2}$ *if* $\max(p,2) < p_0,\ t < \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$.

Proof. *Step 1.* Estimate from below. We divide this step into some cases.
Substep 1.1. The case $p_0 \leq p$. We have

$$b_{D_\mu}(id^*_\mu : (s^{t,\Omega}_{p_0,2}f)_\mu \to (s^{0,\Omega}_{p,2}f)_\mu) \quad = \inf_{\lambda \in (s^{t,\Omega}_{p_0,2}f)_\mu, \lambda \neq 0} \frac{\|\lambda|(s^{0,\Omega}_{p,2}f)_\mu\|}{\|\lambda|(s^{t,\Omega}_{p_0,2}f)_\mu\|}. \tag{4.57}$$

Since $p_0 \leq p$, Lemma 4.25 (i) yields

$$\|\lambda|(s^{t,\Omega}_{p_0,2}f)_\mu\| \lesssim 2^{t\mu}\|\lambda|(s^{0,\Omega}_{p,2}f)_\mu\|$$

for all $\lambda \in (s^{t,\Omega}_{p_0,2}f)_\mu$. Inserting this into (4.57) we find the desired estimate.
Substep 1.2. The case $p \leq p_0 \leq 2$. The proof is similar to Step 1 in the proof of Theorem 4.47 since

$$b_{[D_\mu/2]}(id^{D_\mu}_{2,p}) \asymp x_{[D_\mu/2]}(id^{D_\mu}_{2,p}) \asymp (D_\mu)^{\frac{1}{p}-\frac{1}{2}}\,,$$

if $p \leq 2$, see (4.9c).
Substep 1.3. The case $p \leq 2 < p_0$, $t > \frac{1}{p_0}$. To obtain the lower estimate in this case we combine Lemma 4.28 with (4.24) and (4.9b). The argument follows analogously to Step 1 in the proof of Theorem 4.42.

Substep 1.4. The case $2 \le p < p_0$, $t > \frac{1/p-1/p_0}{p_0/2-1}$. This time we use Lemma 4.28, (4.24) and (4.9a). We choose $\varepsilon > 0$ such that $p \le p_0 - \varepsilon$ and then follow the arguments as in Step 1 of the proof of Theorem 4.40 to obtain the desired estimate.
Substep 1.5. The case $p \le 2 < p_0$ and $t < \frac{1}{p_0}$ can be treated as in Step 2 of the proof of Proposition 4.43, see Lemmas 4.28 and 4.32.
Substep 1.6. The case $2 \le p < p_0$ and $t < \frac{1/p-1/p_0}{p_0/2-1}$. In this case we make use of the embeddings

$$s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f \to s^{0,\Omega}_{2,2}f. \tag{4.58}$$

Property (c) of s-numbers implies

$$b_n(id : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{2,2}f) \lesssim b_n(id : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f).$$

Now Substep 1.5 results in the desired assertion.
Step 2. Estimate from above. The polynomial behaviour of the Weyl numbers of the embedding $id^* : s^{t,\Omega}_{p_0,2}f \to s^{0,\Omega}_{p,2}f$, Theorems 4.41, 4.42 and 4.45, together with Corollary 4.9 result in the upper estimate in the cases $\max(2,p) < p_0$, i.e., parts (ii), (iii). The upper bound in (i) is obtained by applying Proposition 4.48 and Lemma 4.10. The proof is complete. ■

Theorem 4.51. *Let $1 \le p_0 \le \infty$, $1 \le p < \infty$ and $t > \big(\frac{1}{p_0} - \frac{1}{p}\big)_+$. Then*

$$b_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \asymp n^{-\alpha}(\log n)^{(d-1)\beta}, \quad n \ge 2,$$

where

(i) $\alpha = t,\ \beta = \big(t - \frac{1}{p_0} + \frac{1}{2}\big)_+$ *if* $p_0, p \le 2,\ t \ne \frac{1}{p_0} - \frac{1}{2}$ *or* $p_0 \le 2 \le p$;

(ii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{\max(p,2)}$ *if* $\max(p,2) < p_0,\ t > \frac{1/\max(p,2)-1/p_0}{p_0/2-1}$;

(iii) $\alpha = \frac{tp_0}{2},\ \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $p > 1,\ \max(p,2) < p_0,\ t < \frac{1/\max(p,2)-1/p_0}{p_0/2-1}$.

Proof. *Step 1.* Estimate from below. We divide this step into some cases.
Substep 1.1. The case $1 \le p_0, p \le 2$. This proof copies exactly Step 1 of the proof of Theorem 4.46 because of

$$b_n(id^{2n}_{p_0,p}) \asymp x_n(id^{2n}_{p_0,p}) \asymp n^{\frac{1}{p}-\frac{1}{p_0}}$$

if $1 \le p_0, p \le 2$, see (4.9c) and Lemma 4.28.
Substep 1.2. The case $p_0 \le 2 \le p$. From Lemma 4.28 and (4.21) with $\gamma = 2$ we have

$$b_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})} b_n(id^{D_\mu}_{p_0,2}).$$

In view of (4.9c) with $n = [D_\mu/2]$ we obtain

$$b_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \gtrsim 2^{\mu(-t+\frac{1}{p_0}-\frac{1}{2})}(2^\mu \mu^{d-1})^{\frac{1}{2}-\frac{1}{p_0}} = 2^{-\mu t}\mu^{(d-1)(\frac{1}{2}-\frac{1}{p_0})}.$$

Rewrite this in dependence of n we get the claimed estimate in this case.
Substep 1.3. Proof of (ii). The lower estimate in this case is a direct consequence of the results in Theorem 4.50 (ii) and Lemma 4.36.

Substep 1.4. Proof of (iii). If $1 < p \le 2 < p_0$ the argument follows exactly as in Step 1 in the proof of Proposition 4.43 since

$$b_n(id^m_{p_0,p}) \gtrsim m^{\frac{1}{p}-\frac{1}{p_0}}, \quad n \le [m^{\frac{2}{p_0}}],$$

see Lemmas 4.16, 4.17 and 4.28. Employing this results with $p = 2$ and (4.58) we obtain the estimate from below for the case $2 \le p \le p_0$ as well.
Step 2. Estimate from above. We shall use the inequality

$$b_n \lesssim \min(x_n, e_n),$$

see Corollary 4.9 and Lemma 4.10, where we take into account the polynomial behaviour of the Weyl numbers of the embedding $s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f$. The proof is complete. ∎

Remark 4.52. In a view of the proof we find that part (ii) in Theorem 4.51 still holds true if $p_0 = p > 2$. The exact order of asymptotic behaviour of Bernstein numbers in the case $2 < p_0 < p < \infty$ remains open. Here we can give an estimate from below and above

$$n^{-t}(\log n)^{(d-1)t} \lesssim b_n(id^* : s^{t,\Omega}_{p_0,p_0}f \to s^{0,\Omega}_{p,2}f) \lesssim n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \quad n \ge 2.$$

4.5 Transfer to function spaces on the unit cube

4.5.1 The Littlewood-Paley case

The results for Weyl and Bernstein numbers which we obtained in the previous section can be transfered to the level of function spaces. The heart of the matter consists in the following well-known lemma.

Lemma 4.53. *Let $0 < p_0 \le \infty$, $0 < p < \infty$, $q \in \{p_0, 2\}$ and $t \in \mathbb{R}$. Then*

$$\omega_n(id^* : s^{t,\Omega}_{p_0,q}f \to s^{0,\Omega}_{p,2}f) \asymp \omega_n\big(id : S^t_{p_0,q}F(\Omega) \to S^0_{p,2}F(\Omega)\big)$$

holds for all $n \in \mathbb{N}$.

Proof. We follow [140, Section 4.5] and consider the commutative diagram

$$\begin{array}{ccccc}
S^t_{p_0,q}F(\Omega) & \xrightarrow{\mathcal{E}_d} & S^t_{p_0,q}F(\mathbb{R}^d) & \xrightarrow{\mathcal{W}} & s^{t,\Omega}_{p_0,q}f \\
{\scriptstyle id}\big\downarrow & & & & \big\downarrow{\scriptstyle id^*} \\
S^0_{p,2}F(\Omega) & \xleftarrow{R_\Omega} & S^0_{p,2}F(\mathbb{R}^d) & \xleftarrow{\mathcal{W}^*} & s^{0,\Omega}_{p,2}f
\end{array}$$

where $\mathcal{E}_d$ is the extension operator, see Section 1.3, and the mapping $\mathcal{W}$ is defined as

$$\mathcal{W}f := \left\{2^{|\nu|_1}\,\langle f,\, \Psi_{\nu,m}\rangle\right\}_{\nu\in\mathbb{N}_0^d,\, m\in A^\Omega_\nu},$$

see (4.11) for the definition of A^Ω_ν. Furthermore, $\mathcal{W}^*$ is given by

$$\mathcal{W}^*\lambda := \sum_{\nu\in\mathbb{N}_0^d}\sum_{m\in A^\Omega_\nu} \lambda_{\nu,m}\,\Psi_{\nu,m}$$

and R_Ω means the restriction to Ω. From the boundedness of $\mathcal{E}_d, \mathcal{W}, \mathcal{W}^*, R_\Omega$ and property (c) of the s-numbers we obtain $\omega_n(id) \lesssim \omega_n(id^*)$. A similar argument with a slightly modified diagram yields $\omega_n(id^*) \lesssim \omega_n(id)$ as well. ∎

We are now in position to formulate our main results of this section.

Theorem 4.54. *Let $0 < p_0 \le \infty$, $1 < p < \infty$ and $t > \big(\frac{1}{p_0} - \frac{1}{p}\big)_+$. Then we have*

$$x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}\,(\log n)^{(d-1)\,\beta}\,, \qquad n \ge 2$$

where

(i) $\alpha = t,\ \beta = \big(t - \frac{1}{p_0} + \frac{1}{2}\big)_+$ *if* $p_0, p \le 2,\ t \ne \frac{1}{p_0} - \frac{1}{2}$;

(ii) $\alpha = t - \frac{1}{\max(p_0,2)} + \frac{1}{p},\ \beta = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $\max(p_0, 2) \le p$;

(iii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{\max(p,2)}$ *if* $\max(p, 2) < p_0,\ t > \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$;

(iv) $\alpha = \frac{tp_0}{2},\ \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $\max(p, 2) < p_0,\ t < \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$.

In the situation of tensor product Sobolev spaces we have the following.

Theorem 4.55. *Let $1 < p_0, p < \infty$ and $t > \big(\frac{1}{p_0} - \frac{1}{p}\big)_+$. Then we have*

$$x_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}\,, \quad n \ge 2,$$

where

(i) $\alpha = t$ *if* $p_0, p \le 2$;

(ii) $\alpha = t - \frac{1}{\max(2,p_0)} + \frac{1}{p}$ *if* $\max(p_0, 2) \le p$;

(iii) $\alpha = t - \frac{1}{p_0} + \frac{1}{\max(p,2)}$ *if* $\max(p, 2) < p_0,\ t > \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$;

(iv) $\alpha = \frac{tp_0}{2}$ *if* $\max(p, 2) < p_0,\ t < \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$.

Remark 4.56. Theorems 4.54 and 4.55 give the final answer about the behaviour of the Weyl numbers in almost all cases. Only in the limiting cases we are not able to characterize the behaviour of the $x_n(id)$. In this situation we only can give the estimate from above and below, see e.g., [74]. However it is important to mention, that the problem of finding the right order of the s-numbers of embedding of function spaces of dominating mixed smoothness in the limiting cases are more difficult. Even in the isotropic spaces these problems remain open.

Proof of Theorems 4.54, 4.55. The claims in Theorems 4.54 and 4.55 are consequences of Lemma 4.53 and the results in Section 4.4.1. ■

Concerning Bernstein numbers we have the following.

Theorem 4.57. *Let $1 \le p_0 \le \infty$, $1 < p < \infty$ and $t > \big(\frac{1}{p_0} - \frac{1}{p}\big)_+$. Then*

$$b_n(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\beta}, \quad n \ge 2,$$

where

(i) $\alpha = t,\ \beta = \big(t - \frac{1}{p_0} + \frac{1}{2}\big)_+$ *if* $p_0, p \le 2,\ t \ne \frac{1}{p_0} - \frac{1}{2}$ *or* $p_0 \le 2 \le p$;

(ii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{\max(p,2)}$ *if* $\max(p,2) < p_0,\ t > \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$;

(iii) $\alpha = \frac{tp_0}{2},\ \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $\max(p,2) < p_0,\ t < \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$.

Theorem 4.58. *Let* $1 < p_0, p < \infty$ *and* $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$. *Then we have*

$$b_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) \asymp n^{-\beta}(\log n)^{(d-1)\beta}\,, \quad n \geq 2,$$

where

(i) $\beta = t$ *if* $p_0 \leq p$ *or* $p \leq p_0 \leq 2$;

(ii) $\beta = t - \frac{1}{p_0} + \frac{1}{\max(p,2)}$ *if* $\max(p,2) < p_0,\ t > \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$;

(iii) $\beta = \frac{tp_0}{2}$ *if* $\max(p,2) < p_0,\ t < \frac{1/\max(p,2) - 1/p_0}{p_0/2-1}$.

Remark 4.59. Part (ii) in Theorem 4.57 still holds true if $2 < p_0 = p$ and $t > 0$. The case $2 < p_0 < p < \infty$ is left open, see Remark 4.52. Observe that, up to some limiting situations, we have the exact asymptotic behaviour of the Bernstein numbers of the embedding $id : S^t_{p_0}H(\Omega) \to L_p(\Omega)$. Theorems 4.55, 4.58 and Proposition 4.48 show that

$$\begin{aligned} &b_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) \\ &\qquad \asymp \min\{x_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)),\ e_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega))\}. \end{aligned}$$

Except the case $2 < p_0 < p < \infty$ we also have similar estimates in the situation of tensor product Besov spaces. We wish to emphasize that the above equivalence also holds in the case of embeddings $id : A^t_{p_0,q}(\Omega) \to L_p(\Omega)$ with $1 \leq p_0, p, q \leq \infty$ and $t > d(\frac{1}{p_0} - \frac{1}{p})_+$.

Proof of Theorems 4.57, 4.58. Taking into account Lemma 4.53, Theorems 4.50 and 4.51 the claims in Theorems 4.57 and 4.58 follow. ■

To close this section, let us compare the difference of Weyl and Bernstein numbers of the embedding $id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)$ in an $(1/p_0; 1/p)$-plane.

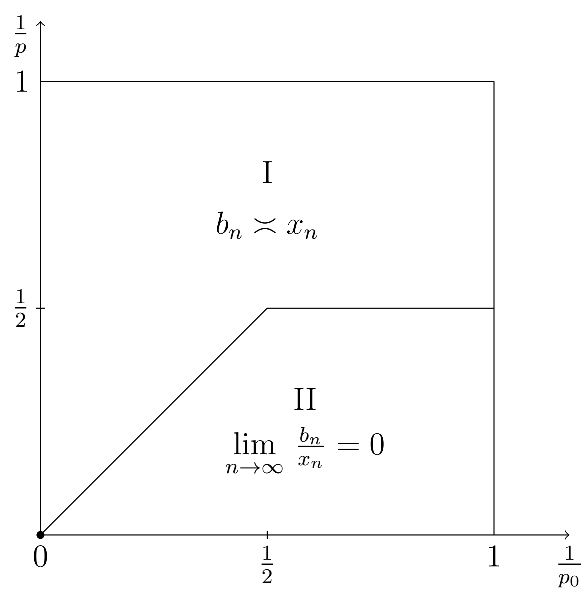

Figure 5. Comparison of Bernstein and Weyl numbers

Figure 5 explains the different behaviour of Bernstein and Weyl numbers. Bernstein numbers are essentially smaller than Weyl numbers in region II, i.e., $\max(p_0, 2) < p$, and they show a similar behaviour in region I. We have the same picture in the case of the embedding $id : S^t_{p_0}H(\Omega) \to L_p(\Omega)$. Note that in our particular situation Bernstein numbers are dominated by Weyl numbers but in general, Bernstein and Weyl numbers are incomparable, see [89].

4.5.2 The extreme cases

Now we turn to extreme cases given by either $p = \infty$ or $p = 1$. Since the decomposition does not work in these cases, we need a different technique. We begin with the results for the space $S^t_{2,2}B(\Omega)$. Recall that $S^t_{2,2}B(\Omega) = S^t_2H(\Omega)$ in the sense of equivalent norms.

Proposition 4.60. (i) *Let $t > \frac{1}{2}$. Then we have*

$$x_n(id: \, S^t_{2,2}B(\Omega) \to L_\infty(\Omega)) \asymp n^{-t+\frac{1}{2}}(\log n)^{(d-1)t}\,, \qquad n \geq 2\,.$$

(ii) *Let $t > 0$. Then we have*

$$x_n(id: \, S^t_{2,2}B(\Omega) \to L_1(\Omega)) \asymp n^{-t}(\log n)^{(d-1)t}\,, \qquad n \geq 2\,.$$

Remark 4.61. For a proof we refer to [121] for part (i) and [99] for part (ii), but see also [20]. In [121, 99, 20] the authors deal with approximation numbers. However, $S^t_{2,2}B(\Omega)$ is a Hilbert space so approximation, Gelfand and Weyl numbers are equal, see Theorem 4.5. Hence, the result in Proposition 4.60 also holds for Gelfand numbers.

By using specific properties of Weyl numbers we will extent Proposition 4.60 to the following result.

Theorem 4.62. *Let $0 < p_0 \leq \infty$.*
(i) *If $t > \frac{1}{p_0}$, then we have*

$$\begin{aligned} x_n(id: \, S^t_{p_0,p_0}B(\Omega) &\to L_\infty(\Omega)) \\ &\asymp \begin{cases} n^{-t+\frac{1}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \text{if } 0 < p_0 \leq 2\,,\ t > \frac{1}{p_0}\,; \\ n^{-t+\frac{1}{p_0}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \text{if } 2 < p_0 \leq \infty\,,\ t > \frac{1}{2}+\frac{1}{p_0}\,; \end{cases} \end{aligned}$$

for all $n \geq 2$.
(ii) *If $t > (\frac{1}{p_0} - 1)_+$, then*

$$x_n(id: S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)) \asymp \begin{cases} n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})_+} & \text{if } p_0 \leq 2,\ t \neq \frac{1}{p_0}-\frac{1}{2}, \\ n^{-t+\frac{1}{p_0}-\frac{1}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \text{if } 2 < p_0 \leq \infty,\ t > \frac{1}{p_0}, \\ n^{-\frac{tp_0}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \text{if } 2 < p_0 < \infty,\ t < \frac{1}{p_0}, \end{cases}$$

for all $n \geq 2$.

Remark 4.63. Recall that $S^t_{p_0,p_0}B(\Omega)$ is compactly embedded into $L_\infty(\Omega)$ if and only if $t > 1/p_0$. The case $2 < p_0 \leq \infty$ and $\frac{1}{p_0} < t \leq \frac{1}{2} + \frac{1}{p_0}$ in part (i) remains open. Observe that part (i) is not the limit of part (ii) in Theorem 4.54 when $p \to \infty$. More exactly, there is a jump of order $(\log n)^{(d-1)/2}$ as it happens many times in this field. The picture in part (ii) is almost complete except the limiting cases.

Proof. *Step 1.* Proof of (i).
Substep 1.1. Estimate from above. Under the given restrictions there always exists some $r > \frac{1}{2}$ such that $t > r + \big(\frac{1}{p_0} - \frac{1}{2}\big)_+$. We consider the continuous embeddings

$$S^t_{p_0,p_0}B(\Omega) \xrightarrow{id_2} S^r_{2,2}B(\Omega) \xrightarrow{id_1} L_\infty(\Omega)\,.$$

The multiplicativity of the Weyl numbers yields

$$x_{2n-1}(id:\, S^t_{p_0,p_0}B(\Omega) \to L_\infty(\Omega)) \le x_n(id_2)\, x_n(id_1)\,.$$

By the lifting property of Besov spaces of dominating mixed smoothness, see Theorem 1.31 (also Lemma 4.37), and $S^0_{2,2}B(\Omega) = L_2(\Omega)$ we have

$$\begin{aligned} x_n(id_2) &\asymp x_n(id : S^{t-r}_{p_0,p_0}B(\Omega) \to L_2(\Omega)) \\ &\asymp \begin{cases} n^{-t+r}(\log n)^{(d-1)(t-r-\frac{1}{p_0}+\frac{1}{2})} & \text{if } \ 0 < p_0 \le 2\,,\ t-r > \frac{1}{p_0} - \frac{1}{2}\,, \\ n^{-t+r+\frac{1}{p_0}-\frac{1}{2}}(\log n)^{(d-1)(t-r-\frac{1}{p_0}+\frac{1}{2})} & \text{if } \ 2 < p_0 \le \infty\,,\ t-r > \frac{1}{p_0}\,. \end{cases} \end{aligned}$$

Now, employing

$$x_n(id_1) \asymp n^{-r+\frac{1}{2}}(\log n)^{(d-1)r}, \qquad n \ge 2,$$

see Proposition 4.60, and the monotonicity of the Weyl numbers the claim follows for all $n \ge 2$.
Step 1.2. Estimate from below. Again we shall use the multiplicativity of the Weyl numbers, but this time in connection with its relation to the 2-summing norm, see Lemma 4.14. We have

$$\begin{aligned} x_{2n-1}&(id : S^t_{p_0,p_0}B(\Omega) \to L_2(\Omega)) \\ &\le x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_\infty(\Omega))\, x_n(id : L_\infty(\Omega) \to L_2(\Omega)) \\ &\le x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_\infty(\Omega))\, n^{-1/2}\, \pi_2(id :\ L_\infty(\Omega) \to L_2(\Omega)) \\ &= x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_\infty(\Omega))\, n^{-1/2}\,; \end{aligned}$$

where in the last equality we have used that

$$\pi_2(id : L_\infty(\Omega) \longrightarrow L_2(\Omega)) = \|id : L_\infty(\Omega) \longrightarrow L_2(\Omega)\| = 1\,,$$

see [87, Example 1.3.9]. Theorem 4.54 shows that

$$\begin{aligned} n^{\frac{1}{2}}\, x_{2n-1}&(id : S^t_{p_0,p_0}B(\Omega) \to L_2(\Omega)) \\ &\asymp \begin{cases} n^{-t+\frac{1}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \text{if } \ 0 < p_0 \le 2\,,\ t > \frac{1}{p_0} - \frac{1}{2}\,, \\ n^{-t+\frac{1}{p_0}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \text{if } \ 2 < p_0 \le \infty\,,\ t > \frac{1}{p_0}\,, \end{cases} \end{aligned}$$

which implies the estimate from below.
Step 2. Proof of (ii).
Substep 2.1. Estimate from above. Since $t > (\frac{1}{p_0} - 1)_+$, there exists $\varepsilon > 0$ such that $t > (\frac{1}{p_0} - \frac{1}{1+\varepsilon})_+$. We consider the continuous embeddings

$$S^t_{p_0,p_0}B(\Omega) \to L_{1+\varepsilon}(\Omega) \to L_1(\Omega)\,.$$

From property (c) of the s-number we obtain

$$x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)) \lesssim x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_{1+\varepsilon}(\Omega))$$

This together with Theorem 4.54 is enough to prove the upper bound.
Substep 2.2. Estimate from below. The chain of continuous embeddings

$$B^{td}_{p_0,p_0}(\Omega) \to S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)\,, \tag{4.59}$$

see Theorem 2.8, and property (c) of the s-numbers yield

$$x_n(id : B^{td}_{p_0,p_0}(\Omega) \to L_1(\Omega)) \lesssim x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega))\,.$$

For the behaviour of Weyl numbers in the situation of isotropic spaces we refer to Pietsch [85], Lubitz [65] and Caetano [13, 14, 15]. In this case we have

$$x_n(id : B^{td}_{p_0,p_0}(\Omega) \to L_1(\Omega)) \asymp n^{-t}\,, \qquad n \in \mathbb{N}$$

if $p_0 \le 2$, $\frac{t}{d} > (\frac{1}{p_0} - 1)_+$. This implies the lower estimate for the case $p_0 < 2$, $t < \frac{1}{p_0} - \frac{1}{2}$. Next we prove that

$$x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)) \gtrsim n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}$$

if $p_0 \le 2$ and $t > \frac{1}{p_0} - \frac{1}{2}$. This time we employ the interpolation property of the Weyl numbers. There always exists a pair (Θ, p) such that $0 < \Theta < 1\,,\ 1 < p < 2$ and

$$\|f|L_p(\Omega)\| \le \|f|L_1(\Omega)\|^{1-\Theta}\,\|f|L_2(\Omega)\|^{\Theta} \qquad \text{for all} \quad f \in L_2(\Omega). \tag{4.60}$$

Now Theorem 4.11 yields

$$\begin{aligned} x_{2n-1}&(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \\ &\lesssim x_n^{1-\Theta}(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega))\; x_n^{\Theta}(id : S^t_{p_0,p_0}B(\Omega) \to L_2(\Omega)). \end{aligned}$$

Note that $0 < p_0 \le 2$ and $t > \frac{1}{p_0} - \frac{1}{2}$ imply

$$\begin{aligned} x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_2(\Omega)) &\asymp x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \\ &\asymp n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \end{aligned}$$

see Theorem 4.54. This leads to

$$x_n(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)) \gtrsim n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}.$$

The lower bounds in the remaining cases can be treated similarly. We finish the proof. ■

The above argument can be applied in the situation of tensor product Sobolev spaces. We have the following theorem.

Theorem 4.64. *Let* $1 < p_0 < \infty$. *Then we have*
(i)

$$x_n(id :\ S^t_{p_0}H(\Omega) \to L_\infty(\Omega)) \asymp \begin{cases} n^{-t+\frac{1}{2}}(\log n)^{(d-1)t} & \textit{if} \quad p_0 \le 2\,,\ t > \frac{1}{p_0}, \\ n^{-t+\frac{1}{p_0}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} & \textit{if} \quad 2 < p_0\,,\ t > \frac{1}{p_0} + \frac{1}{2}, \end{cases}$$

(ii) *and*

$$x_n(id:\ S^t_{p_0}H(\Omega)\to L_1(\Omega)) \asymp \begin{cases} n^{-t}(\log n)^{(d-1)t} & \textit{if}\ \ p_0\le 2\,,\ t>0,\\ n^{-t+\frac{1}{p_0}-\frac12}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac12)} & \textit{if}\ \ 2<p_0,\ t>\frac{1}{p_0},\\ n^{-\frac{tp_0}{2}}(\log n)^{(d-1)\frac{tp_0}{2}} & \textit{if}\ \ 2<p_0,\ t<\frac{1}{p_0}, \end{cases}$$

for all $n\ge 2$.

We now discuss the asymptotic behaviour of Bernstein numbers of embeddings of Sobolev and Besov spaces into $L_1(\Omega)$. By using Corollary 4.9 and Theorems 4.62, 4.64 we obtain the upper estimate.

Proposition 4.65. (i) *Let* $1\le p_0\le\infty$ *and* $t>0$*. Then*

$$b_n(id:S^t_{p_0,p_0}B(\Omega)\to L_1(\Omega)) \lesssim \begin{cases} n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac12)_+} & \textit{if}\ p_0\le 2,\ t\ne\frac{1}{p_0}-\frac12,\\ n^{-t+\frac{1}{p_0}-\frac12}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac12)} & \textit{if}\ 2<p_0\le\infty,\ t>\frac{1}{p_0},\\ n^{-\frac{tp_0}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac12)} & \textit{if}\ 2<p_0<\infty,\ t<\frac{1}{p_0}, \end{cases}$$

for all $n\ge 2$.
(ii) *Let* $1<p_0<\infty$ *and* $t>0$*. Then*

$$b_n(id:\ S^t_{p_0}H(\Omega)\to L_1(\Omega)) \lesssim \begin{cases} n^{-t}(\log n)^{(d-1)t} & \textit{if}\ \ p_0\le 2\,,\ t>0,\\ n^{-t+\frac{1}{p_0}-\frac12}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac12)} & \textit{if}\ \ 2<p_0,\ t>\frac{1}{p_0},\\ n^{-\frac{tp_0}{2}}(\log n)^{(d-1)\frac{tp_0}{2}} & \textit{if}\ \ 2<p_0,\ t<\frac{1}{p_0}, \end{cases}$$

for all $n\ge 2$.

In some situations we have exact asymptotic order.

Theorem 4.66. (i) *Let* $1<p_0<2$*. Then*

$$b_n(id:\ S^t_{p_0,p_0}B(\Omega)\to L_1(\Omega)) \asymp \begin{cases} n^{-t} & \textit{if}\quad 0<t<\frac{1}{p_0}-\frac12,\\ n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac12)} & \textit{if}\quad t>\frac12, \end{cases}$$

for all $n\ge 2$.
(ii) *Let* $1<p_0\le 2$ *and* $t>0$*. Then*

$$b_n(id:\ S^t_{p_0}H(\Omega)\to L_1(\Omega)) \asymp n^{-t}(\log n)^{(d-1)t},\qquad n\ge 2.$$

Proof. We focus on the lower estimate.
Step 1. Proof of (i). By employing the chain of embeddings (4.59), and property (c) of the s-numbers we obtain

$$b_n(id:B^{td}_{p_0,p_0}(\Omega)\to L_1(\Omega)) \lesssim b_n(id:S^t_{p_0,p_0}B(\Omega)\to L_1(\Omega))\,.$$

Now the claim for the case $0<t<\frac{1}{p_0}-\frac12$ follows from

$$b_n(id:B^{td}_{p_0,p_0}(\Omega)\to L_1(\Omega)) \asymp n^{-t}\qquad n\in\mathbb{N}$$

if $1 < p_0 \leq 2$, $t > 0$, see [75]. To prove for the case $t > \frac{1}{2}$ we employ the interpolation property of the Bernstein numbers (in connection with Gelfand numbers), see Theorem 4.13. From (4.60) we have

$$\begin{aligned} & b_{2n-1}(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \\ & \qquad \lesssim b_n^{1-\Theta}(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega))\ c_n^{\Theta}(id : S^t_{p_0,p_0}B(\Omega) \to L_2(\Omega)), \end{aligned}$$

for $1 < p < 2$ and $\Theta \in (0, 1)$. Take into account

$$b_{2n-1}(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})},$$

see Theorem 4.57, and

$$c_n(id : S^t_{p_0,p_0}B(\Omega) \to L_2(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})},$$

if $t > \frac{1}{2}$, see Theorem 4.80, we conclude the desired estimate in this case.

Step 2. Proof of (ii). By using Theorems 4.58, 4.79 and a similar argument as Step 1 we obtain (ii) as well. ■

4.5.3 Comparison with known results

In this section we shall compare the results of Bernstein numbers in Theorem 4.58 with those obtained by Galeev [40]. Galeev in [40] studied the behaviour of Bernstein numbers of embeddings of periodic Sobolev spaces with bounded mixed derivative defined on d-dimensional torus $\mathbb{T}^d$ into Lebesgue spaces. By using the Nikol'skij duality theorem, he reduced the estimation of Bernstein numbers to the calculation of Kolmogorov numbers of the adjoint operator. He obtained the following.

Theorem 4.67 (Galeev)**.** *Let $1 < p_0, p < \infty$ and $t > (\frac{1}{p_0} - \frac{1}{p})_+$. Then*

$$b_n(id : S^t_{p_0}H(\mathbb{T}^d) \to L_p(\mathbb{T}^d)) \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}, \quad n \geq 2,$$

where

(i) $\alpha = t$ *if* $p_0 \leq p < 2,\ t > \frac{1}{2}$ *or* $p \leq p_0 \leq 2$ *or* $p_0 \leq p$;

(ii) $\alpha = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $p \leq 2 \leq p_0,\ t > \frac{1}{p_0}$;

(iii) $\alpha = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $2 \leq p \leq p_0,\ t > \frac{1}{p_0} - \frac{1}{p} + \frac{1}{2}$.

Comparing Theorem 4.67 with Theorem 4.58 we found that the behaviour of Bernstein numbers in both settings coincide. However, in the cases $p_0 \leq p < 2$ and $2 \leq p \leq p_0$ Geleev was using some additional smoothness. He also was unable to determine the asymptotic behaviour of Bernstein numbers in the cases of low smoothness. In the same paper Galeev obtained some results for Bernstein numbers of embedding of Nikol'skij spaces.

Theorem 4.68 (Galeev)**.** *Let $1 < p_0, p < \infty$. Then*

$$b_n(id : S^t_{p_0,\infty}B(\mathbb{T}^d) \to L_p(\mathbb{T}^d)) \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}, \quad n \geq 2,$$

where

(i) $\alpha = t$ *if* $p_0, p < 2,\ t > \frac{1}{2}$;

(ii) $\alpha = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $p \le 2 \le p_0,\ t > \frac{1}{p_0}$;

(iii) $\alpha = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $2 \le p \le p_0,\ t > \frac{1}{p_0} - \frac{1}{p} + \frac{1}{2}$.

Observe that the picture given in Theorem 4.68 is not complete. In this situation Galeev also could not determine the asymptotic behaviour of Bernstein numbers in the case $2 < p_0 < p < \infty$, see Remark 4.52.

4.6 Applications of Weyl and Bernstein numbers

The particular interest in Weyl numbers stems from the fact that they are the smallest known s-numbers satisfying the famous Weyl-type inequalities. Let $T : X \to X$ be a compact linear operator in a Banach space X and $\{\lambda_n(T)\}_{n=1}^{\infty}$ be the sequence of non-zero eigenvalues of T, ordered in the following way: each eigenvalue is repeated according to its algebraic multiplicity and $|\lambda_n(T)| \ge |\lambda_{n+1}(T)|$, $n \in \mathbb{N}$. Then the inequality

$$|\lambda_{2n-1}(T)| \le \sqrt{2e}\Big(\prod_{k=1}^{n} x_k(T)\Big)^{1/n}, \tag{4.61}$$

holds for all $n \in \mathbb{N}$, see Pietsch [85] and Carl, Hinrichs [17]. This inequality should be compared with the Carl-Triebel inequality which states

$$|\lambda_n(T)| \le \sqrt{2}\, e_n(T)\,, \tag{4.62}$$

see Carl, Triebel [19] (see also [18, 33]). Hence, Weyl and entropy numbers are tools to estimate the behaviour of eigenvalues of compact linear operators. There are good reasons to compare Weyl numbers with entropy numbers of the embedding $id : S^t_{p_0}H(\Omega) \to L_p(\Omega)$. In view of Theorem 4.55 and Proposition 4.48 we have

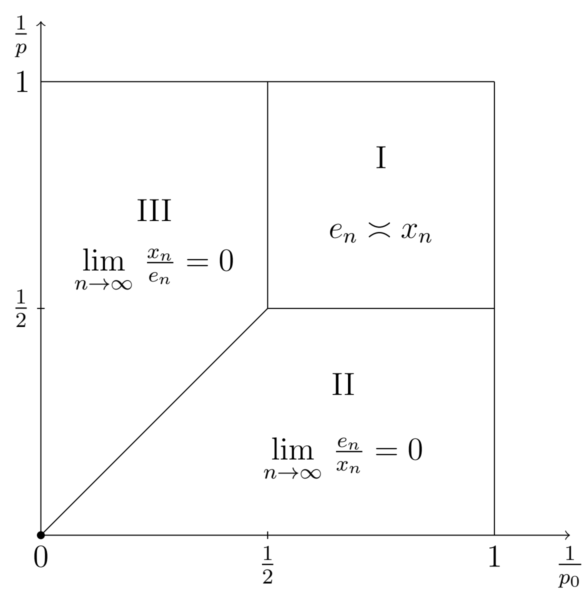

Figure 6. Comparison of Weyl and entropy numbers.

We use Figure 6 to explain the different behaviour of entropy and Weyl numbers of the embedding $id : S^t_{p_0}H(\Omega) \to L_p(\Omega)$. Weyl numbers are essentially smaller than entropy numbers in region III, entropy numbers are essentially smaller than Weyl numbers in region II, and they show a similar behaviour in region I. In the case of tensor product Besov spaces we have a similar picture.

Based on the results in Theorem 4.55 and Proposition 4.48 we are able to control the eigenvalues of some compact operators $T : L_p(\Omega) \to L_p(\Omega)$. We have the following theorem.

Theorem 4.69. *Let $1 < p_0, p < \infty$ and $t > (\frac{1}{p_0} - \frac{1}{p})_+$. Let further $T : L_p(\Omega) \to L_p(\Omega)$ be a compact operator. Assume that T can be decomposed $T = id \circ A$ where $A : L_p(\Omega) \to S^t_{p_0}H(\Omega)$ is a linear bounded operator and $id : S^t_{p_0}H(\Omega) \to L_p(\Omega)$. Then there exists a constant $C > 0$ such that*

$$|\lambda_n(T)| \le C\|A\| \cdot n^{-\alpha}(\log n)^{(d-1)\alpha}$$

where

(i) $\alpha = t$ *if* $p_0 \le p$;

(ii) $\alpha = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $(p < p_0 \le 2$ *or* $2 \le p < p_0)$ *and* $t > \frac{1/p - 1/p_0}{\max(p_0,p')/2 - 1}$;

(iii) $\alpha = t + \max(\frac{1}{p} - \frac{1}{2}, \frac{1}{2} - \frac{1}{p_0})$ *if* $p \le 2 \le p_0,\ t > \frac{1}{\max(p_0,p')}$;

(iv) $\alpha = \frac{t \max(p_0,p')}{2}$ *if* $p \le 2 \le p_0$ *and* $t < \frac{1}{\max(p_0,p')}$ *or*
if $(p < p_0 \le 2$ *or* $2 \le p < p_0)$ *and* $t < \frac{1/p - 1/p_0}{\max(p_0,p')/2 - 1}$.

Proof. Firstly, from inequalities (4.61), (4.62) and properties of Weyl and entropy numbers we have

$$\begin{aligned} |\lambda_{2n-1}(T)| &\le \min\Big\{\sqrt{2}\, e_{2n-1}(T), \sqrt{2e}\Big(\prod_{k=1}^n x_k(T)\Big)^{1/n}\Big\} \\ &\le \|A\| \cdot \min\Big\{\sqrt{2}\, e_{2n-1}(id), \sqrt{2e}\Big(\prod_{k=1}^n x_k(id)\Big)^{1/n}\Big\}. \end{aligned} \tag{4.63}$$

Let T^*, A^* and id^* be dual operators of T, A and id respectively. Since T is a compact operator we have similar inequality

$$|\lambda_{2n-1}(T)| = |\lambda_{2n-1}(T^*)| \le \|A^*\| \cdot \min\Big\{\sqrt{2}\, e_{2n-1}(id^*), \sqrt{2e}\Big(\prod_{k=1}^n x_k(id^*)\Big)^{1/n}\Big\}. \tag{4.64}$$

The lifting property of function spaces of dominating mixed smoothness, see Theorem 1.31, and property (c) of Weyl and entropy numbers lead to

$$s^*_n(id^* : L_{p'}(\Omega) \to S^{-t}_{p'_0}H(\Omega)) \asymp s^*_n(id_1 : S^t_{p'}H(\Omega) \to L_{p'_0}(\Omega)).$$

Here s^* are either entropy numbers or Weyl numbers. This together with Theorem 4.55 and Proposition 4.48 implies

$$e_n(id^* : L_{p'}(\Omega) \to S^{-t}_{p'_0}H(\Omega)) \asymp n^{-t}(\log n)^{(d-1)t}$$

and

$$x_n(id^* : L_{p'}(\Omega) \to S^{-t}_{p'_0}H(\Omega)) \asymp n^{-\beta}(\log n)^{(d-1)\beta}$$

where

$$\beta = \begin{cases} t & \text{if } p', p'_0 \le 2 \\ t - \frac{1}{\max(2,p')} + \frac{1}{p'_0} & \text{if } \max(p',2) \le p'_0 \\ t - \frac{1}{p'} + \frac{1}{\max(p'_0,2)} & \text{if } \max(p'_0,2) < p',\ t > \frac{1/\max(p'_0,2)-1/p'}{p'/2-1} \\ \frac{tp'}{2} & \text{if } \max(p'_0,2) < p',\ t < \frac{1/\max(p'_0,2)-1/p'}{p'/2-1}. \end{cases}$$

From the polynomial behaviour of $x_n(id)$ and $x_n(id^*)$ and the fact that $\|A\| = \|A^*\|$ we have from (4.63), (4.64)

$$|\lambda_n(T)| \le C \cdot \|A\| \cdot \min\{e_n(id), e_n(id^*), x_n(id), x_n(id^*)\}.$$

Comparing $e_n(id)$, $e_n(id^*)$, $x_n(id)$ and $x_n(id^*)$ the assertion follows. ■

Remark 4.70. A counterpart of Theorem 4.69 in the isotropic setting can be found in [65, Satz 4.17] and [55, Proposition 3.c.10]. As an example of an operator A in Theorem 4.69 we may use the tensor product Riemann-Liouville operator given by

$$\mathcal{R}^d_\alpha f := (\mathcal{R}_\alpha \otimes \ldots \otimes \mathcal{R}_\alpha) f \quad (d - \text{folds})$$

for $f \in L_1(\Omega)$ where $\alpha > 0$ and

$$\mathcal{R}_\alpha h(\xi) := \frac{1}{\Gamma(\alpha)} \int\limits_0^\xi (\xi - s)^{\alpha-1}\, h(s)\ \mathrm{d}s\,, \quad h \in L_1([0,1]), \qquad \xi \in [0,1]\,.$$

If $\alpha = 1$, $\mathcal{R}_\alpha$ is actually the Volterra operator. It is obvious that if $f \in L_p(\Omega)$ and $\alpha \in \mathbb{N}$ then $\mathcal{R}^d_\alpha$ is a continuous mapping from $L_p(\Omega)$ into $S^\alpha_p W(\Omega)$. Approximation and entropy numbers of the tensor product Riemann-Liouville operator have been considered by Kühn and Linde [60] with motivation traced back to optimal series representations of the fractional Brownian sheet.

Remark 4.71. We wish to recall another application of Weyl numbers in the isotropic situation. Pietsch [86, 87] (in one dimension) and König [54, 55, 56] have used the behaviour of Weyl numbers $x_n(id : B^t_{p,q}(\Omega) \to L_s(\Omega))$ to estimate eigenvalues of the compact integral operator

$$T_K f(x) = \int\limits_\Omega K(x,y) f(y) \mathrm{d}y$$

in a Lebesgue space on Ω, where the kernel $K(x,y)$ belongs to $B^{t_1}_{p_1,q_1}(\Omega, B^{t_2}_{p_2,q_2}(\Omega))$, $x, y \in \Omega$. They showed that if $t_1, t_2 > 0$ and $t_1 + t_2 > d\big(\frac{1}{p_1} + \frac{1}{p_2} - 1\big)$ then the eigenvalues of T_K belong to a certain Lorentz sequence space. One may follow their argument to extend the result to dominating mixed spaces.

Remark 4.72. A further application of Weyl numbers is to serve as lower bound for Gelfand numbers and approximation numbers. For an example, let us refer to Theorem 4.82 in which Weyl numbers are sharp lower bounds for approximation and Gelfand numbers of embedding of tensor product Sobolev and Besov spaces in the sup-norm. It is worth pointing out that the study of approximation of functions with mixed smoothness in the uniform norm (L_∞-norm) is more difficult. We refer to comments and open problems presented in the survey [29, Sections 4.5 and 4.6].

Remark 4.73. In contrast to Weyl numbers, Bernstein numbers serve as lower bounds for Kolmogorov, Gelfand and entropy numbers, see Theorem 4.5 and Lemma 4.10. Beside that, Bernstein numbers are lower bounds for non-linear n-widths. We refer to [22], but see also [27, 30]. Especially, Bernstein numbers are lower bounds for the error analysis of Monte-Carlo algorithms, see Kunsch [59].

4.7 Asymptotic behaviour of some other s-numbers

As we have noticed in Section 4.1, Weyl and Bernstein numbers are closely related to some other s-numbers, namely approximation, Kolmogorov and Gelfand numbers. Hence, this section is devoted to recall the asymptotic behaviour of those numbers of embedding of tensor product Besov and Sobolev spaces. There are many contributions dealing with the asymptotic behaviour of Kolmogorov, Gelfand and approximation numbers of the embedding $id:\ S^t_{p_0,q}A(\Omega) \to L_p(\Omega)$. We refer to Bazarkhanov [7, 8], Kühn, Sickel, Ullrich [61] and Cobos, Kühn, Sickel [20] for the most recent publication in this direction. The topic itself has been investigated at various places over the last 30 years, see, e.g., Belinsky [9], Galeev [39, 41, 42], Romanyuk [91] - [99] and Temylakov [115, 117, 120, 121, 122], Especially we refer to the recent survey [29] which contains an almost exhaustive collection of known facts about s-numbers of the identities $id:\ S^t_{p_0,q}A(\Omega) \to L_p(\Omega)$. We begin with the results about approximation numbers.

Theorem 4.74. *Let $1 < p_0, p < \infty$ and $t > (\frac{1}{p_0} - \frac{1}{p})_+$. Then we have*

$$a_n(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\beta}\,, \quad n \geq 2,$$

where

(i) $\alpha = t,\ \beta = t + (\frac{1}{2} - \frac{1}{p_0})_+$ *if* $p \leq p_0$;

(ii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $p_0 \leq p \leq 2$ *or* $2 \leq p_0 \leq p$;

(iii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $2 < p < p_0'$, $t > \frac{1}{p_0}$;

(iv) $\alpha = \beta = t - \frac{1}{2} + \frac{1}{p}$ *if* $2 \leq p_0' < p$, $t > 1 - \frac{1}{p}$;

(v) $\alpha = \frac{p_0'}{2}(t - \frac{1}{p_0} + \frac{1}{p})$, $\beta = \frac{2\alpha}{p_0'}$ *if* $2 \leq p_0' < p$, $t < 1 - \frac{1}{p}$.

Remark 4.75. Parts (i)-(iv) have been proved by Romanyuk [95, 96] and Bazarkhanov [7, 8]. Part (v) is recently obtained in [77].

Theorem 4.76. *Let $1 < p_0, p < \infty$ and $t > \big(\frac{1}{p_0} - \frac{1}{p}\big)_+$. Then we have*

$$a_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}\,, \quad n \geq 2,$$

where

(i) $\alpha = t - \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$ *if* $p_0 \geq 2$ *or* $p \leq 2$;

(ii) $\alpha = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $2 < p \leq p_0'$, $t > \frac{1}{p_0}$;

(iii) $\alpha = t - \frac{1}{2} + \frac{1}{p}$ *if* $2 < p_0 \leq p$, $t > 1 - \frac{1}{p}$.

Concerning the asymptotic behaviour of Kolmogorov numbers we have the following.

Theorem 4.77. *Let* $1 \leq p_0 \leq \infty$, $1 < p < \infty$ *and* $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$. *Then*

$$d_n(id : S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\beta}, \quad n \geq 2,$$

where

(i) $\alpha = t,\ \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $\max(2,p) \leq p_0$ *or* $2 \leq p_0 < p$, $t > \frac{1/p_0 - 1/p}{1-2/p}$;

(ii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $1 < p_0 < p \leq 2$;

(iii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $p_0 \leq 2 < p$, $t > \frac{1}{p_0}$.

Theorem 4.78. *Let* $1 < p_0, p < \infty$ *and* $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$. *Then we have*

$$d_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}, \quad n \geq 2,$$

where

(i) $\alpha = t - \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$ *if* $p \leq p_0$ *or* $p_0 \leq p \leq 2$;

(ii) $\alpha = t - (\frac{1}{p_0} - \frac{1}{2})_+$ *if* $\max(p_0, 2) < p$, $t > \max(\frac{1}{p_0}, \frac{1}{2})$.

Based on the duality of Kolmogorov and Gelfand numbers, see [84, Theorem 11.7.7] and the lifting property of spaces of dominating mixed smoothness, see Theorem 1.31 and also Lemma 4.37, we obtain

$$\begin{aligned} c_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) &= d_n(id : L_{p'}(\Omega) \to S^{-t}_{p_0'}H(\Omega)) \\ &\asymp d_n(id : S^t_{p'}H(\Omega) \to L_{p_0'}(\Omega)). \end{aligned}$$

In view of Theorem 4.78 we have the following result.

Theorem 4.79. *Let* $1 < p_0, p < \infty$ *and* $t > \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$. *Then we have*

$$c_n(id : S^t_{p_0}H(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\alpha}, \quad n \geq 2,$$

where

(i) $\alpha = t - \left(\frac{1}{p_0} - \frac{1}{p}\right)_+$ *if* $p \leq p_0$ *or* $2 \leq p_0 \leq p$;

(ii) $\alpha = t - (\frac{1}{2} - \frac{1}{p})_+$ *if* $p_0 \leq \min(p, 2)$, $t > \max(\frac{1}{2}, 1 - \frac{1}{p})$.

Observe, to obtain the asymptotic behaviour of Gelfand numbers of the embedding $id: S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)$ by applying the same duality argument as in Theorem 4.79 we would need to know

$$d_n(id: \; S^t_{p'}H(\Omega) \to S^0_{p_0',p_0'}B(\Omega)),$$

since $[S^t_{p_0,p_0}B(\Omega)]' = S^{-t}_{p_0',p_0'}B(\Omega)$, see Proposition 1.45. However, these numbers are not investigated except in a few special cases, e.g., $p_0 = 2$. By the same method for Weyl numbers in Sections 4.3 and 4.4 we obtain the following results, for more details see [77].

Theorem 4.80. *Let* $1 < p_0, p < \infty$ *and* $t > (\frac{1}{p_0} - \frac{1}{p})_+$. *Then we have*

$$c_n(id: \; S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)) \asymp n^{-\alpha}(\log n)^{(d-1)\beta}, \qquad n \geq 2\,,$$

where

(i) $\alpha = t,\ \beta = t - \frac{1}{p_0} + \frac{1}{2}$ *if* $\max(2,p) \leq p_0$ *or* $\big(p_0, p < 2,\ t > \frac{1}{2}\big)$;

(ii) $\alpha = \beta = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $2 \leq p_0 \leq p$;

(iii) $\alpha = t - \frac{1}{2} + \frac{1}{p},\ \beta = t - \frac{1}{p_0} + \frac{1}{p}$ *if* $p_0 < 2 \leq p,\ t > 1 - \frac{1}{p}$;

(iv) $\alpha = \frac{p_0'}{2}(t - \frac{1}{p_0} + \frac{1}{p}),\ \beta = \frac{2\alpha}{p_0'}$ *if* $\big(p_0 \leq 2 < p,\ t < 1 - \frac{1}{p}\big)$
or $\big(p_0 < p \leq 2,\ t < \frac{1/p_0 - 1/p}{2/p_0 - 1}\big)$.

Let us make a comparison between approximation and Gelfand numbers of the embedding $id: S^t_{p_0,p_0}B(\Omega) \to L_p(\Omega)$. The difference between these numbers is illustrated in the following figure, see Theorems 4.74 and 4.80. We assume the dimension $d \geq 2$.

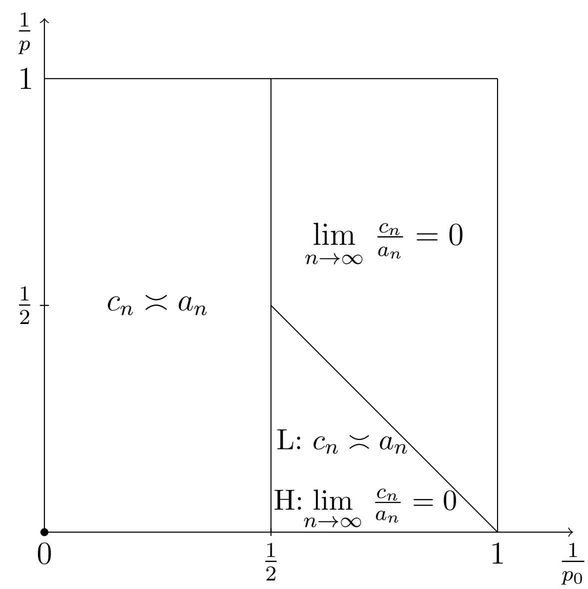

Figure 7. Comparison of approximation and Gelfand numbers

Here H refers to the domain of "high smoothness", i.e., $t > 1 - \frac{1}{p}$ and L refers to "low smoothness", i.e., $t < 1 - \frac{1}{p}$. Figure 7 indicates that Gelfand numbers and approximation numbers show similar behaviour if either $p_0 \geq 2$ or $2 \leq p_0' < p$, $t < 1 - \frac{1}{p}$, i.e., $c_n \asymp a_n$. In other cases Gelfand number are essentially smaller than approximation numbers, i.e., $\lim_{n\to\infty} \frac{c_n}{a_n} = 0$.

We wish to mention that Gelfand and approximation numbers play a crucial role in information-based complexity. In fact, Gelfand and approximation numbers are inversely related to the information complexity. For more details we refer to the monographs [80, 126].

To finish this section, we collect some results about the asymptotic behaviour of approximation, Kolmogorov and Gelfand numbers in the extreme cases, i.e., $p = 1$ or $p = \infty$. For the following we refer to [99] and [77].

Theorem 4.81. (i) *Let $2 \leq p_0 < \infty$ and $t > 0$. Then we have*

$$a_n(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \qquad n \geq 2\,.$$

(ii) *Let $1 < p_0 < \infty$ and $t > 0$. Then we have*

$$a_n(id : S^t_{p_0}H(\Omega) \to L_1(\Omega)) \asymp n^{-t}(\log n)^{(d-1)t}, \qquad n \geq 2\,.$$

(iii) *Let either $2 \leq p_0 < \infty$ and $t > 0$ or $1 < p_0 < 2$ and $t > \frac{1}{2}$. Then we have*

$$c_n(id : S^t_{p_0,p_0}B(\Omega) \to L_1(\Omega)) \asymp n^{-t}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})}, \qquad n \geq 2\,.$$

(iv) *Let $1 < p_0 < \infty$ and $t > 0$. Then we have*

$$c_n(id : S^t_{p_0}H(\Omega) \to L_1(\Omega)) \asymp n^{-t}(\log n)^{(d-1)t}, \qquad n \geq 2\,.$$

Since $L_\infty(\Omega)$ has the metric extension property, see [84, Proposition C.3.2.2] and also [90, page 36], we have $c_n(T) = a_n(T)$ for any linear bounded operator T from Banach spaces X into $L_\infty(\Omega)$, see [84, Proposition 11.5.3].

Theorem 4.82. *Let $1 < p_0 \leq 2$ and $t > 1$. Then we have*

$$\begin{aligned} c_n(id : S^t_{p_0,p_0}B(\Omega) \to L_\infty(\Omega)) &= a_n(id : S^t_{p_0,p_0}B(\Omega) \to L_\infty(\Omega)) \\ &\asymp n^{-t+\frac{1}{2}}(\log n)^{(d-1)(t-\frac{1}{p_0}+\frac{1}{2})} \end{aligned}$$

and

$$\begin{aligned} c_n(id : S^t_{p_0}H(\Omega) \to L_\infty(\Omega)) &= a_n(id : S^t_{p_0}H(\Omega) \to L_\infty(\Omega)) \\ &\asymp n^{-t+\frac{1}{2}}(\log n)^{(d-1)t}, \end{aligned}$$

for all $n \geq 2$.

Remark 4.83. Recall that Theorem 4.82 still holds true if $p_0 = 2$ and $t > \frac{1}{2}$. Beside the above results in the extreme cases we refer to Temlyakov [122] where he obtained the asymptotic behaviour of Kolmogorov numbers in the two-dimensional situation only. The problem remains open in higher dimensions. For some results on Kolmogorov numbers with a gap between the estimates from above and below we refer to Romanyuk [98], but see also Belinsky [9].

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